Real Analysis Exchange Vol. 9 (1983-84)

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Transformations of Functions

For the present note, a transformation of a function f will mean a function of the form $h \circ f$ where h is a homeomorphism from the reals to the reals. Clearly, some classes of functions are unchanged by transformations; such as C, A, $\mathcal{DB}_1$ (the classes of continuous, approximately continuous and Baire 1 Darboux functions). That is, $h \circ f$ belongs to the class for each f in the class and each homeomorphism h. If P is a class of functions HP will denote $\{h \circ f : f \in P$ and h is a homeomorphism$\}$. Clearly, every $g \in HP$ can be transformed by a suitable homeomorphism to a function $f \in P$.

To date three non-trivial classes of functions have been characterized. They are:

(in C) 1. $HBV = HBVG^* = HT_1 = B_2$ [1]

(in C) 2. $HAC = H\,Diff = HACG^* = HS = S'$ [6]

(in A) 3. $H\Lambda = Gs.a.c.$ [7]

Here BV, AC, Diff, are the functions of bounded variation, absolutely continuous functions, and differentiable functions (defined on a closed interval); the definitions of BVG*, ACG*, ACG and BVG are standard (cf. [9]). The remaining classes are defined as follows:

T_1: $f \in T_1$ if $|\{y: f^{-1}(y)$ is infinite$\}| = 0$.

B_2: $f \in B_2$ if $\{y: f^{-1}(y)$ is finite$\}$ is c-dense in the range of f .

S : $f \in S$ if for each $\varepsilon > 0$ there is a $\delta > 0$ so that $|E|<\delta$ implies $|f(E)| < \varepsilon$.

S': $f \in S'$ if for each interval J in the range of f there is a $\delta > 0$ so that $|E|<\delta$ implies J is not contained in $f(E)$.

S_1: $f \in S_1$ if for each interval J in the range of f there is a $\delta > 0$ so that $|E|<\delta$ implies $|J \setminus f(E)| > 0$.

L : $f \in L$ if every point x in the domain of f is a Lebesgue point of f .

s.a.c.: f is s.a.c. on a set E provided that for each $\alpha > 0$ there is $T_\alpha > 0$ so that $\lim\limits_{\substack{x \in I \\ |I| \to 0}} \dfrac{|I \cap (|f| \geq T_\alpha)|}{|I|} \leq \alpha$ for each $x \in E$.

Gs.a.c.: $f \in$ Gs.a.c. if f is approximately continuous and every closed subset of the domain of f contains a portion on which f is strongly approximately continuous (s.a.c.)

Since every $\mathcal{DB}_1$ function is equivalent to a Lebesgue integrable derivative under a suitable change of variables (a theorem of Maximoff which has been given a clear proof by Preiss in [8]), it follows that an investigation of what makes a function a derivative must necessarily either look at properties which are invariant under transformations or examine properties which are neither invariant under transformations nor under changes of variables. The symbols $b\Delta$, $L\Delta$, and Δ will denote the classes of bounded derivatives, Lebesgue integrable derivatives and derivatives each defined on [0,1]. It would be valuable to have characterizations of $Hb\Delta$, $HL\Delta$, and $H\Delta$ within the class $\mathcal{D}B_1$. For this purpose, the Denjoy-Clarkson property [5], Zahorski's properties M_3 and M_4 [11] for Δ and $b\Delta$, and Weil's property Z are all invariant under transformations (Cf. also [4]). The purpose here will be to point out some of the anomalies and regularities of the three known classes (new observations which are of value in their own right) and to make comparisons to and observations of the class of transformations of derivatives.

To begin at the beginning, Nina Bary in her investigation [1] of sums of compositions of absolutely continuous functions and sums of compositions of continuous functions of bounded variation came upon a characterization of continuous functions in HBV. She denoted this by B_2 (defined above). It is clear that B_2 is preserved by all transformations. To see that the property is right is a two step process. First, if f is continuous and satisfies B_2, then there is a homeomorphism so that

$g = h \circ f \in T_1$. This homeomorphism expands the set of points y where $f^{-1}(y)$ is finite into a set of full measure in the range of f . Second, if g is continuous and satisfies T_1 , there is an absolutely continuous homeomorphism h so that $h \circ g$ is of bounded variation. The function h is defined by

$$h(y) = \int_c^y 1/s_g(t)\, dt$$

where $[c, d]$ is the range of g and

$s_g(y)$ = the cardinality of $\{x \mid f(x) = y\}$

The combined transformation of f to a continuous function of bounded variation involves a property of what homeomorphisms can do and a second 'mechanical' homeomorphism.

A similar two step process is involved with the class HDiff. If f is continuous and satisfies S' , then f can be transformed to a Lipschitz function by a homeomorphism. This homeomorphism is defined 'mechanically' in terms of f by

$$h_1(y) = \begin{cases} 0 & \text{if } y = c \text{ where } [c, d] \text{ is the range of } f \\ \inf \{t : \exists E \text{ with } f(E) \supset [c, y] \text{ and } t = |E|\} . \end{cases}$$

That $g = h_1 \circ f$ satisfies a Lipschitz condition with a single Lipschitz constant is a straightforward calculation using only that f is continuous. Then h_2 is defined so as to be differentiable and so that $\{y : h'_2(y) = 0\}$ contains the image under g of the set of points where g is not differentiable. Since this image set is of measure 0, there are homeomorphisms with this property (cf. [11]). Since S' is HS', since $S \subset S'$ and since a continuous function is in S iff it is of the form $h \circ f$ where h is a homeomorphism and both h and f are absolutely continuous (cf. [1] or [9] p. 296-289), the characterization of $HAC = S'$ is also established.

Finally, in order that a function be transformed into L, it is necessary that it be approximately continuous. The property G s.a.c. is preserved by transformations. Let E_n be the sets on which f is s.a.c. Then, for each natural number k there is a positive number $T_n(k)$ and $\delta_k > 0$ such that $|I \cap (|f| \geq T_n(k))| \leq k^{-3}|I|$ whenever $x \in I \cap E_n$ and $|I| < \delta_k$. Clearly, the $T_n(k)$ can be chosen such that $T_n(k) \to \infty$ as $k \to \infty$. The functions $T_n(k)$ give rise to 'mechanically' to homeomorphisms which transform each of the countable sets E_n on which f is s.a.c. Specifically, if $h(x) / x \to 0$ as $x \to \infty$, and if h_n is an odd homeomorphism with $h_n(T_n(k)) < k - 1$, for $k > 1$, $h \circ h_n$ transforms f to a function whose Lebesgue points contain E_n. Then f is transformed by a homeomorphism which is odd and eventually smaller than each of the homeomorphisms $h \circ h_n$ (cf. [7]).

While HL is a subset of $H\Delta$, it is a particularly nice subset. Specifically, it is a uniformly closed algebra of functions and if $I = \cup E_n$ and $f \in HL$ on E_n then $f \in HL$ (cf. 7) .

The class H Diff is not so nice. In fact, if f is defined to be a strictly increasing singular function from $[0, \frac{1}{2}]$ to $[0, 1]$ and is defined on $[\frac{1}{2}, 1]$ by $f(x) = 2 - 2x$, then f can be transformed on either half of the interval (by its inverse) but can not be transformed on the unit interval because it takes a set of measure 0 onto its range.

It is tempting then to try to find a property or properties which guarantee that f can be transformed on the union of two or finitely many intervals whenever f can be transformed on each of them. If $f \in S$ on I_2 and if, on I_1 , $f \in S_1$, then $f \in S'$ on $I_1 \cup I_2$. Clearly, S_1 is necessary on I_1 for transforming each function f on $I_1 \cup I_2$ when $f \in S$ on I_2 . Define the property S_1' by "f satisfies S_1' iff for each interval J contained in the range of f , there is a $\delta > 0$ so that $|E|<\delta$ implies $|J \setminus f(E)| > \delta$. It is clear that S_1' is sufficient on I_1 for a function satisfying S on I_2 to be transformable on $I_1 \cup I_2$.

Theorem 1. $S_1 \equiv S_1'$ (The proof is given at the end of the paper.) One can further play with this notion and define S_p , $p \in (0, 1)$ by $f \in S_p$ iff for each J contained in the range of f , $\exists\, \delta > 0$ so that $|E| < \delta$ implies $|f(E) \cap J| < p\,|J|$. If

$f \in S_{p_i}$ on I_i and $\Sigma p_i < 1$, then $f \in S'$ on $\cup I_i$. Here we find ourselves 'driven back' to condition S. In fact,

Theorem 2. $S = S_p$ (The proof is again given at the end of the paper.) Oddly enough, the same type of phenomenon occurs for continuous functions in B_2. This is a consequence of the following theorem:

Theorem 3. *If E is a dense G_δ set in $[c, d]$ with $H = E^c$ also dense, then there is a continuous $f : [0, 1]$ onto $[c, d]$ such that $\{y : f^{-1}(y)$ is perfect $\supset E$ and, except for an at most countable set of points of H, $f^{-1}(y)$ is finite for $y \in H$.* (The proof is again deferred.)

Thus, it is possible to define a function f which is B_2 on I_1 and on I_2 but not on $I_1 \cup I_2$. For example, let E_1 be a G_δ subset of $[0, 1]$ of measure 1 whose complement is c-dense and E_2 be a G_δ subset of measure 0 containing E_1^c. Define f on I_1 and $I_2 = I_3 \cup I_4$ so that $f^{-1}(y)$ has uncountably many preimages in $I_1(I_4)$ for each $y \in E_1$ (E_2) and f is linear on I_3 and continuous on $I_1 \cup I_2$. If $f^{-1}(y)$ is finite in I_1 (I_4) at all but an at most countable set of points of ${E^c}_1({E^c}_2)$, then $f \in B_2$ on I_1 and I_2 but not on $I_1 \cup I_2$.

It turns out that the class $H\Delta$ and, in fact, the class $Hb\Delta$ exhibit the same type of phenomena; namely, there are functions in HbΔ on I_1 and on I_2 which are not in HbΔ on $I_1 \cup I_2$. It is possible to use Bruckner's 'inflexible derivatives' (cf. [2] or [3]) to show this. (This can be done by transforming an inflexible derivative at $0 = f(0)$ on $[0, \frac{1}{2}] = I_1$ with a homeomorphism h which is not symmetric in any internal containing the origin (with $h(0) = 0$) . If the transformed function is then extended on $[\frac{1}{2}, 1] = I_2$ to a inflexible derivative at 1 , the resulting function will not be transformable on $I_1 \cup I_2$ even though it is on I_1 and on I_2 .) However, the following example is simpler and will also be used to illuminate HbΔ later.

__Example 1__. __There exists__ f __on__ [0, 1] __such that__ f $\in$ HbΔ __on each interval__ [0, 1-ε] __and__ [ε, 1] __but__ f $\notin$ HbΔ __on__ [0, 1] .

__Construction.__

$$\text{Let } g(x) = \begin{cases} 0 & \text{if } x = 0 , \frac{2}{3} \text{ or } 1 \\ 1 & \text{if } x = \frac{1}{2} \text{ or } \frac{5}{6} \\ -1 & \text{if } x = \frac{1}{6} \end{cases}$$

and extend g linearly on the intervals between these points. On $[0, \frac{1}{2}]$, let $f(x) = g(n(n+1)(x - \frac{1}{n+1}))$ for $x \in [\frac{1}{n+1}, \frac{1}{n}]$ $n > 1$. Let $f(0) = 0$ and on $(\frac{1}{2}, 1)$ let $f(x) = -f(1-x)$. Pictorially speaking, f has two upward saw teeth for each downward one on $[0, \frac{1}{2}]$ and the reverse on $[\frac{1}{2}, 1]$. It is easy to check that

$$h_1(x) = \begin{cases} x & \text{if } x \geq 0 \\ 2x & \text{if } x \leq 0 \end{cases}$$

transforms f on $[0, 1-\varepsilon]$ and that

$$h_2(x) = \begin{cases} x & \text{if } x \le 0 \\ 2x & \text{if } x \ge 0 \end{cases} \qquad \text{transforms } f$$

on $[\varepsilon, 1]$.

Let h be any homeomorphism with $h(0) = 0$. Then

$$\int_0^{1/k} h(kx)\, dx = \frac{1}{k} \int_0^1 h(x)\, dx \quad \text{and} \quad \int_{-\frac{1}{k}}^0 h(kx)\, dx = \frac{1}{k} \int_{-1}^0 h(x)\, dx .$$

Suppose that $h \circ f \in \Delta$. Then on $I_n = [0, \frac{1}{n}]$,

$$n \int_{I_n} h \circ f = \frac{2}{3} \int_0^1 h + \frac{1}{3} \int_{-1}^0 h \quad \text{and on} \quad I'_n = [1- \frac{1}{n}, 1]$$

$$n \int_{I'_n} h \circ f = \frac{1}{3} \int_0^1 h + \frac{2}{3} \int_{-1}^0 h . \quad \text{Thus if as } n \to \infty ,$$

$$n \int_{I_n} h \circ f \to 0 , \text{ then } \int_0^1 h = - \frac{1}{2} \int_{-1}^0 h . \quad \text{Consequently as}$$

$n \to \infty$, $n \int_{I'_n} h \circ f = - \int_{-1}^0 h$ does not approach 0. Since there is no loss in assuming that $h(0) = 0$, if $h \circ f$ is a derivative at 0, it is not a derivative at 1.

Note that all of the associated sets of f are M_4 sets. (HbΔ cannot be characterized by associated sets). Further, if

$$f_n(x) = \begin{cases} (1 + \frac{1}{n})\, f(x) & \text{if } x \in [0, \frac{1}{2}] \\ f(x) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}, \quad \text{Then}$$

each $f_n \in Hb\Delta$ and $f_n \to f$ uniformly. Thus, while Δ is closed under uniform limits, $Hb\Delta$ is not. However, if $f \in Hb\Delta$ and $c(x)$ is a bounded approximately continuous function which is 0 at each point x where f is not approximately continuous, then $f(x) + c(x) \in Hb\Delta$. In fact the same homeomorphism h which transforms $f(x)$ transforms $f(x) + c(x)$. This is because

$$\int_I \frac{h(f(x) + c(x))\, dx}{|I|} = \int_I \frac{h(f(x))}{|I|} + \int_I \frac{h(f(x)) + c(x)) - h(f(x))}{|I|}$$

and the last term tends to 0 as $I \to x_o$ at each x_o where $c(x_o) = 0$, due to the uniform continuity of h .
At all other points $f(x) + c(x)$ is approximately continuous and hence $h(f(x) + c(x))$ is the derivative of its integral.

The $f(x)$ in example 1 is not in $Hb\Delta$, but the proof relies heavily on the triangular character of the graph of f and the equal heights of the triangles. Thus it is not clear as to whether a function slightly different from f might be in $Hb\Delta$. A principle for distinguishing is given below. It is based on a very simple fact; namely, if $h_1(x) \geq h_2(x)$ on an interval I and $h_1(x) > h_2(x)$ on a set of positive measure of I , then given any homeomorphism h , $\int_I h \circ h_1 > \int_I h \circ h_2$.

This is, of course, because equality would imply that

$h \circ h_1 = h \circ h_2$ a.e.

Given any g measurable and defined on $[0, 1]$, let

$g^{\alpha}(x) = \sup \{y : |(g > y)| > x\}$. (This follows [9, p. 144])

Given any f measurable and defined on $I = [a, b]$, let

$g_I(x) = f(t)$ if $x = (t-a)/(b-a)$. Let $h_I(x) = g_I{}^{\alpha}(x)$.

Note that $g^{\alpha}(x)$ is a nondecreasing function which is 'equimeasurable' with $g(x)$; i.e., $|(g^{\alpha} < y)| = |(g < y)|$.

Thus $\int_0^1 g^{\alpha}(x)\, dx = \int_0^1 g(x)\, dx$. Note further that if f is Lebesgue integrable, then for each interval I

$$\int_I \frac{f(t)}{|I|}\, dt = \int_0^1 g_I(x)\, dx = \int_0^1 h_I(x)\, dx$$

where $h_I(x)$ is non-decreasing on $[0, 1]$. Then for homeomorphisms h which preserve the Lebesgue integrability of f ,

$$* \quad \int_I \frac{h \circ f(t)\, dt}{|I|} = \int_0^1 h \circ h_I(x)\, dx .$$

This gives rise to the following criteria for determining that a function $f \notin HL\Delta$. (These criteria work e.g. on f of example 1.)

1. If there are two sequences of intervals I_n and J_n so that $I_n \to x_o$, $J_n \to x_1$, $f(x_o) = f(x_1)$ (x_o may equal x_1) such that $h_{I_n} \geq h_{J_n}$ and if there is a set E with $|E| > 0$

on which $h_{I_n} > h_{J_n}$, then f can not be transformed. (This is because of the 'simple' principle and the relationship*)

2. If there are two sequences of intervals I_n and J_n so that $I_n \to x_o$, $J_n \to x_1$, $f(x_o) = f(x_1)$ such that $h_{I_n} \geq h_{J_n}$ and if there is an $\varepsilon > 0$ and sets E_n with $|E_n| > \varepsilon$ on which $h_{I_n} \geq h_{J_n} + \varepsilon$, then f can not be transformed. (This is because $|E| \geq \varepsilon$ where $E = \overline{\lim}\, E_n$. Then 1 applies for a subsequence of the h_{I_n}.)

3. If there is a sequence of intervals I_n and for each I_n a sequence J_{nm} with $I_n \to x_o$, $J_{nm} \to x_m$ and $f(x_m) = f(x_o)$ for all m and $h_{I_n} \geq h_{J_{nm}}$ for all m and n and there is an $\varepsilon > 0$ so that for all n and m there is a set E_{nm} so that $h_{I_n} \geq h_{J_{nm}} + \varepsilon$ on E_{nm}, then f cannot be transformed (for similar reasons).

4. If there are sequences I_n and J_n with $I_n \to x_o$, $J_n \to x_1$ and $f(x_o) > f(x_1)$ and $h_{I_n} \leq h_{J_n}$, then f cannot be transformed.

(This is because to transform such an f consists of finding an increasing homeomorphism h such that

$$\int_{I_n} \frac{h \circ f}{|I_n|} \to h(f(x_o)) \quad \text{and} \quad \int_{I_n} \frac{h \circ f}{|J_n|} \to h(f(x_1)) .$$

but $\displaystyle\int_{I_n} \frac{h \circ f}{|I_n|} = \int_0^1 h \circ h_{I_n} \leq \int_0^1 h \circ h_{J_n} = \int_{J_n} \frac{h \circ f}{|J_n|}$.)

It is not known to the author whether these principles characterize the bounded $\mathcal{D}B_1$ functions which satisfy the Denjoy-Clarkson Property and have associated sets in class M_4 but cannot be transformed into bounded derivatives. The principles have the property that they are preserved under transformations and can fail to hold on two intervals I_1 and I_2 while holding on $I_1 \cup I_2$. It would be interesting to see a function for which the principles do not suffice.

Proofs of Theorems 1, 2, and 3.

1. Clearly S'_1 implies S_1 . Suppose S_1 holds and S'_1 does not. Then there is an interval J contained in the range of f and $\delta_0 > 0$ so that f does not take any set E with $|E| < \delta_0$ onto almost all of J . There are also sets E_n with $|E_n| < \delta_n = \delta_0 \cdot 2^{-n}$ such that $|J \setminus f(E_n)| \leq \delta_n$. But then $E = \cup E_n$ satisfies $|E| < \delta_0$. However $|J \setminus f(E)| = 0$, a contradiction.
2. Clearly $S \subset S_p$. Suppose there is a continuous $f \in S_p \setminus S$. Let $\varepsilon > 0$ be given and suppose $\exists\, E_n$ with $|E_n| < 2^{-n}$ and $|f(E_n)| \geq \varepsilon$. Since f is continuous, the sets $f(E_n)$ and E_n can be chosen to be compact. Letting

$B = \overline{\lim}\, f(E_n)$, $|B| \geq \varepsilon$ and, if $A_n = \bigcup_{n}^{\infty} E_k$, $B \subset f(A_n)$ for each natural number n . Let G be an open set with $G \supset B$ and with $|B| > p\,|G|$. Then there is a component interval J of G such that $|B \cap J| > p\,|J|$. But then $|f(A_n) \cap J| > p\,|J|$ for each n and since $|A_n| \to 0$, this contradicts the fact that $f \in S_p$.

3. Given a rectangle $R = [a, b] \times [c, d]$ and an open set $G \subset [c, d]$ let $f_R\,(a) = c$, $f_R\,(\frac{2a+b}{3}) = d$, $f_R\,(\frac{a+2b}{3}) = c$ $f_R\,(b) = d$, $g_R\,(a) = d$, $g_R\,(\frac{2a+b}{3}) = c$, $g_R\,(\frac{a+2b}{3}) = d$, $g_R\,(b) = c$ and define f_R and g_R linearly between these points. Let $\sigma(R, I, G)$ and $\sigma'(R, I, G)$ consist of the graph of f_R (resp. g_R) along with all points belonging to rectangles $\overline{I} \times \overline{J}$ where $J \subset G$ and $f_R(I) = J$ (resp. $g_R[I] = J$). If E is a G_δ subset of $[c, d]$, let G_n be a decreasing sequence of open sets with $E = \cap G_n$. For $R_1 = [0, 1] \times [c, d]$, let $E_1 = \sigma(R_1\,, [0, 1]\,, G_1)$. Replace each rectangle R in E_1 with either $\sigma(R, I, G_2)$ or $\sigma'(R, I, G_2)$, where $R = I \times J$, in such a fashion that the resulting set E_2 is connected. Continuing this from the rectangles of E_n to form the set E_{n+1} results in a sequence of closed sets E_n whose intersection is the graph of a continuous function. It is observable from this

construction that if $x \in F_n = G_n^c$ and x is not an end point of a component interval of G_n, then $f^{-1}(f(x))$ consists of no more than 3^n points. For all x in $\cap G_n$, $f^{-1}(f(x))$ is a perfect set. Thus f satisfies the conclusion of the theorem.

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Received March 23, 1984.