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SOME APPLICATIONS OF PARTITIONING COVERS.

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1. Basic Ideas

Let $I_o = [a, b]$ be a compact interval and let $\mathcal{I}_o$ be the set of all closed sub-intervals of I_o.

A finite subset of $\mathcal{I}_o \times I_o$, $\pi = \{(I_j, x_j), 1 \leq j \leq n\}$ say, is called a PARTITION of $\bigcup_{j=1}^{n}$ provided $\mathring{I}_j \cap \mathring{I}_k = \phi$, $1 \leq j \neq k \leq n$.

Suppose that with all $x \in I_o$ we have associated an $\mathcal{I}_x \subset \mathcal{I}_o$ such that the collection $\tilde{\mathcal{I}} = \{I;\ I \in {}_x,\ x \in I_o\}$ covers I_o then $\tilde{\mathcal{I}}$ is called a COVER OF I_o.

A cover $\tilde{\mathcal{I}}$ is said to be PARTITIONING, or a PARTITIONING COVER, iff $\forall I \in \mathcal{I}_o$ $\exists$ partition $\pi = \{(I_j, x_j), 1 \leq j \leq n\}$, of I for which $I_j \in \mathcal{I}_{x_j}$, $1 \leq j \leq n$; then π is called an $\tilde{\mathcal{I}}$-PARTITION of I.

2. Examples and an Elementary Lemma.

2.1 If $\delta \in \mathbb{R}$, $\delta > 0$ and $\forall x \in I_o$, $\mathcal{I}_x = \{I;\ x \in I,\ |I| < \delta,\ I \in \mathcal{I}_o\}$, then we write $\mathcal{U} = \tilde{\mathcal{I}}$ and call this the UNIFORM COVER of I_o. $\mathcal{U}$ is obviously partitioning and a $\mathcal{U}$-partition of $[c, d] \in \mathcal{I}_o$ can be written $\pi = (a_o, \ldots, a_n;\ x_1, \ldots, x_n)$ where $c = a_o < a_1 < \ldots < a_n = d$, $a_{i-1} \leq x_i \leq a_i$, $a_i - a_{i-1} < \delta$, $1 \leq i \leq n$.

This note owes much to many discussions with Dr. John Upton of the Department of Mathematics, University of Melbourne, and was written while the author was visiting research fellow in the department.

2.2 If $\delta: I_o \to R$, $\delta > 0$ and $\forall x \in I_o$, $\mathcal{J}_x = \{I;\ x \in I,\ |I| < \delta(x),$ $I \in \mathcal{J}_o\}$, then we write $\mathcal{O} = \mathcal{J}$, and call this the ORDINARY-COVER of I_o. A simple compactness argument shows that $\mathcal{O}$ is partitioning; (see Thomson [9,10].) An $\mathcal{O}$-partition of $[c, d] \in \mathcal{J}_o$ can be written as is the $\mathcal{U}$-partition in 2.1 except that now we require that, $a_i - a_{i-1} < \delta(x_i)$, $1 \leq i \leq n$.

Remarks (1): Both $\mathcal{U}$ and $\mathcal{O}$ would be better described as $\mathcal{U}_\delta$ and $\mathcal{O}_\delta$ respectively as they are completely determined once δ is given. Sometimes this is expressed by saying that $\mathcal{U}$ or $\mathcal{O}$ is δ-FINE. Similar comments apply to later examples.

2.3 If $E \subset I_o$, then the LOWER INNER RIGHT DENSITY of E at x is

$$^{+}\underline{\rho}_*(E, x) = \liminf_{u \to x+} \frac{|E \cap [x, u]|_*}{u - x}.$$

Here $|E|_*$ denotes the inner Lebesgue measure of E). In a similar way we can define $^{-}\underline{\rho}_*(E, x)$, the LOWER INNER LEFT DENSITY of E at x.

Suppose that with all $x \in I_o$ we associate a set A_x with $^{-}\underline{\rho}_*(A_x, x) > \lambda$, $^{+}\underline{\rho}_*(A_x, x) > \rho$, with obvious modifications if $x = a$ or $x = b$. For this to be meaningful we clearly require $0 \leq \rho, \lambda < 1$; it is extended to allow ρ and, or, λ to be 1, by then requiring the respective densities to be equal to 1.

Now $\forall x \in I_o$ let $\mathcal{J}_x = \{[u, v];\ u \leq x \leq v,\ u, v \in A_x,\ [u, v] \in \mathcal{J}_o\}$

and then $\tilde{\mathcal{J}} = (\rho, \lambda)\text{-}\mathcal{D}$ is called the (ρ, λ)-DENSITY COVER of I_o. A category argument shows that if $\rho + \lambda > 1$, then $(\rho, \lambda)\text{-}\mathcal{D}$ is partitioning; (see Thomson, [9, 10].) A $(\rho, \lambda)\text{-}\mathcal{D}$-partition of $[c, d] \in \mathcal{J}_o$ can be written as in 2.1 except that now we require tht a_{i-1}, $a_i \in A_{x_i}$, $1 \leq i \leq n$.

The special partitioning covers $(1, 1)\text{-}\mathcal{D}$, and $(\frac{1}{2}, \frac{1}{2})\text{-}\mathcal{D}$ are called the APPROXIMATE, and PREPONDERANT COVERS, respectively. We will write

$$\mathcal{A}p = (1, 1)\text{-}\mathcal{D}, \quad \mathcal{P}r = (\tfrac{1}{2}, \tfrac{1}{2})\text{-}\mathcal{D}.$$

2.4 Suppose that with $x \in I_o$ we associate a filter $\mathcal{F}_x$ that converges to x. Then $\forall x \in I_o$ let $\mathcal{J}_x = \{[u, v];\ u \leq x \leq v,\ u, v \in F,\ F \in \mathcal{F}_x,\ [u, v] \in \mathcal{J}_o\}$. Putting $\mathcal{F} = \tilde{\mathcal{J}}$ we get a FILTER COVER of I_o.

Particular choices of $\{\mathcal{F}_x;\ x \in I_o\}$ give the previous two examples. General properties of the filters $\mathcal{F}_x$, $x \in I_o$ can be given that will imply that $\mathcal{F}$ is partitioning; (see Thomson [9].)

2.5 If in 2.2 we define $\mathcal{J}_x = \{I;\ I \subset]x - \delta(x), x + \delta(x)[,\ I \in \mathcal{J}_o\}$, then $\mathcal{S} = \tilde{\mathcal{J}}$ is called the SHARP COVER of I_o. As for $\mathcal{O}$ it is easily seen that $\mathcal{S}$ is partitioning, and an $\mathcal{S}$-partition of $[c, d] \in \mathcal{J}_o$ can be written as in 2.2 except that now we require

$[a_{i-1}, a_i] \subset]x_i - \delta(x_i), x_i + \delta(x_i)[$, $1 \leq i \leq n$.

2.6 If $\mathcal{J} \subset \mathcal{J}_o$, then $\mathcal{J}$ is said to be (a) ADDITIVE iff when $[c, d]$ and $[d, e]$ are in $\mathcal{J}$ this implies $[c, e] \in \mathcal{J}$; (b) $\tilde{\mathcal{J}}$-LOCAL, $\tilde{\mathcal{J}}$ a cover of

I_o, (as in 1.), iff $x \in I_o$, $\mathcal{J}_x \subset \mathcal{J}$.

The usefulness of partitioning covers is in part due to the following very simple lemma.

Lemma. If $\mathcal{J} \subset \mathcal{J}_o$ is both additive and $\tilde{\mathcal{J}}$-local for some partitioning cover I_o, then $\mathcal{J} = \mathcal{J}_o$.

3. Some Simple Applications

3.1 (Darboux Properties). If f: $I_o \to R$ is continuous and never zero, let $\mathcal{J} = \{[u, v]; [u, v] \in \mathcal{J}_o$ and $f(u)f(v) > 0\}$. Clearly $\mathcal{J}$ is additive and the hypotheses on f imply $\mathcal{J}$ is $\mathcal{O}$-local. Hence by the lemma $\mathcal{J} = \mathcal{J}_o$; i.e. $\forall u, v, a \le u < v \le b, f(u)f(v) > 0$, which is equivalent to saying that f is Darboux.

If we had only assumed f to be approximately continuous, preponderatly continuous, or even just (ρ, λ)-continuous, $\rho + \lambda > 1$, then $\mathcal{J}$ would have been $\mathcal{A}$-, $\mathcal{P}$-, $(\rho, \lambda)\mathcal{D}$-local, respectively and so again, by the lemma, $\mathcal{J} = \mathcal{J}_o$, and f is Darboux.

3.2 (Bolzano-Weierstrass Theorem). Let $A \subset I_o$, $A' = \phi$ and put $\mathcal{J} = \{I; I \in \mathcal{J}_o$ and $I \cap A$ is finite$\}$. Obviously $\mathcal{J}$ is additive and $\mathcal{O}$-local; in fact, $\forall x \in I_o \exists \delta(x) > 0$ such that if $x \in I$, $|I| < \delta(x)$, and if $I \in \mathcal{J}_o$, then either $I \cap A = \phi$ or $I \cap A = \{x\}$. Hence, by the lemma $\mathcal{J} = \mathcal{J}_o$ and so, in particular, $A = A \cap I_o$ is finite.

3.3 (Heine-Borel Theorem). Let $\{G\}$ be an open cover of I_o and define

$\mathcal{J} = \{I;\ I \in \mathcal{J}_o$ and $\exists$ finite subset of $\{G\}$ that covers $I\}$. $\mathcal{J}$ is additive and $\mathcal{O}$-local; in fact $\forall x \in I_o\ \exists\ G \in \{G\}$ with $x \in G$ and so $\exists$ $\delta(x) > 0$ such that $I \in \mathcal{J}$, $x \in I$ and $|I| < \delta(x)$ implies $I \subset G$. Hence by the lemma $\mathcal{J} = \mathcal{J}_o$ and so, in particular, I_o can be covered by a finite subset of $\{G\}$.

3.4 (A weak Vitali Theorem). Let $\mathcal{O}$ be an ordinary full cover of I_o, $E \subset I_o$ and let $\mathcal{E} = \{I\colon I \in \mathcal{J}_x,\ x \in E\}$; $\mathcal{E}$ can be thought of as an ordinary full cover of E.

Given $\varepsilon > 0$ let $I_n \in \mathcal{J}_o$, $n = 1, 2, 3, \ldots$, $E \subset \bigcup_{n=1}^{\infty} \overset{\circ}{I}_n$ and $\sum_{n=1}^{\infty} |I_n| < |E|^* + \varepsilon$; further let $J_n \in \mathcal{J}_o$, $n = 1, 2, \ldots$, $I_o \sim E \subset \bigcup_{n=1}^{\infty} \overset{\circ}{J}_n$ and $\sum_{n=1}^{\infty} |J_n| < |I_o \sim E|^* + \varepsilon$; then $\sum_{n=1}^{\infty} |J_n| < b - a - |E|_* + \varepsilon$.

We now define another full cover $\mathcal{O}'$ of I_o as follows:

if $x \in E$, $\mathcal{J}'_x = \{I;\ I \in \mathcal{J}_x$ and $I \subset \overset{\circ}{I}_n$ for some $n\}$; (0)

if $x \notin E$, $\mathcal{J}'_x = \{I;\ I \in \mathcal{J}_o$, $x \in I$ and $I \subset \overset{\circ}{J}_n$ for some $n\}$.

Let $\pi = (a_o, \ldots, a_n;\ x_1, \ldots, x_n)$ be an $\mathcal{O}'$-partition of I_o: then

$$b - a = \sum_{i=1}^{n} (a_i - a_{i-1}) = \sum_{i \in A} (a_i - a_{i-1}) + \sum_{i \in B} (a_i - a_{i-1}),$$

where

$$A = \{i;\ x_i\ \varepsilon\ E\} \quad \text{and} \quad B = \{i;\ x_i\ \varepsilon\ E\}.$$

Clearly, on the one hand

$$\sum_{i\,\varepsilon A} (a_i - a_{i-1}) \leq \sum_{n-1}^{\infty} |I_n| < |E|^* + \varepsilon,$$

and on the other hand,

$$\sum_{i\,\varepsilon A} (a_i - a_{i-1}) = (b - a) - \sum_{i\,\varepsilon B} (a_i - a_{i-1}) \geq (b - a) - \sum_{n=1}^{\infty} |J_n| > |E|_* - \varepsilon.$$

Thus given any $E \subset I_o$, $\varepsilon > 0$ and any full cover $\mathcal{E}$ of E then $\mathcal{E}$ contains a finite subset $I_1, \ldots, I_n$ with $\overset{\circ}{I}_k \cap \overset{\circ}{I}_j = \phi$, $1 \leq k \neq j \leq n$, such that

$$|E|_* - \varepsilon < \sum_{k=1}^{n} |I_k| < |E|^* + \varepsilon. \tag{1}$$

3.5 (Integrals) Let $f\colon\ I_o \to R$, then we say that f is $\mathcal{O}$-INTEGRABLE on I_o, with $\mathcal{O}$-INTEGRAL EQUAL TO c iff $\forall \varepsilon > 0\ \exists \delta\colon I_o \to R$, $\delta > 0$ such that $\forall \delta$-fine $\mathcal{O}$-partitions (i.e. $v\mathcal{O}_\delta$-partitions, see Remark 1) of I_o, $\pi = (a_o, \ldots, a_n;\ x_1, \ldots, x_n)$, we have that

$$\left| \sum_{i=1}^{n} f(x_i)a_i - a_{i-1}) - c \right| < \varepsilon.$$

In a similar way we can define $\mathcal{R}$-, (ρ, λ)-$\mathcal{P}$-, $(\rho + \lambda) > 1$, and $\mathcal{L}$-INTEGRALS. Each of the integrals defined this way is a well known classical integral: (a) the $\mathcal{R}$-integral is the Riemann integral; (b) the $\mathcal{P}$- integral is the Perron integral; (c) the $\mathcal{L}$-integral is the Lebesgue integral; (d) while the (ρ, λ)-$\mathcal{P}$-integral appears to be new, special cases are well known; the $\mathcal{A}_p$-integral is the Burkill approximatley continuous Perron integral, and the $\mathcal{P}_p$-integral is the preponderantly continuous Perron integral of Izumi; (see Bullen [2], Burkill [1], Henstock [3, 4], Izumi [6], Pfeffer [7] and Thomson [9, 10].)

The elementary properties of all of these integrals follow by simple modifications of the elementary procedures used for the Riemann integral. In all cases except the $\mathcal{R}$-integral, (1) implies that if $|E| = 0$ and $1_E(x) = 1$, $x \varepsilon E$ and $1_E(x) = 0$, $x \varepsilon E$, then 1_E is integrable to 0, and so the extension of the definitions to finite almost everywhere, or even to defined almost everywhere functions, is fairly straightforward.

There is no particular difficulty in extending the above definition to obtain Stieltjes' integrals; (see in particular the above references Henstock, Pfeffer and Thomson.)

4. Monotonicity Theorems

4.1 The lemma above can be used to prove the following elementary monotonicity theorem.

Lemma 1. Let $F: I_o \to R$ be such that $\underline{D}F \geq 0$, then F is increasing.

Proof:
Given $\varepsilon > 0$ let $\mathscr{S} = \{[u, v];\ a \leq u < v \leq b \text{ and } F(v) - F(u) \geq -\varepsilon(v - u)\}$. Clearly $\mathscr{S}$ is additive and $\mathscr{O}$-local and so $\mathscr{S} = \mathscr{S}_o$; ε being arbitrary the result is immediate.

4.2 The object of this section is to see how Lemma 1 can be generalised. Simple examples show that if $\underline{D}F(x) \geq 0$ fails to hold at one point, the result can be false; in the above proof this is because $\mathscr{S}$ fails to be $\mathscr{O}$-local at that point. We will consider various hypotheses that will allow us to define an $\mathscr{S}$ that is $\mathscr{O}$-local. For simplicity we will assume that the point c at which $\underline{D}F(x) < 0$ is neither a nor b; the cases $c = a$, $c = b$ are easily considered.

Lemma 2. If $F: I_o \to R$ is such that $\underline{D}F(x) \geq 0$, $x \neq c$ and

$$\limsup_{h \to 0+} F(c - h) \leq F(c) \leq F(c) \leq \liminf_{h \to 0+} F(c + h),$$

then F is increasing.

Proof: Given $\varepsilon > 0$ we define $\mathscr{S}_x$, $x \in I_o$, as follows:

$x \neq c$:

$$\mathcal{I}_x = \{[u, v];\ a < u < x < v < b, \text{ and } F(v) - F(u) > -\varepsilon(v - u)\}; \tag{2}$$

$$\mathcal{I}_c = \{[u, v];\ a < u < \cancel{x}\,c < v < b, \text{ and } F(v) - F(u) > -\varepsilon\}.$$

Clearly $\tilde{\mathcal{I}} = \{I;\ I \in \mathcal{I}_x,\ x \in I_o\}$ is an $\mathcal{O}$-full cover of I_o. Let $\pi = (a_o, \ldots, \cancel{x_n}\, a_n)$ be an $\tilde{\mathcal{I}}$-partition of $[u, v] \in \mathcal{I}_o$. N/c x_n (3)

$$\text{Then} \qquad F(v) - F(u) = \sum_{i=1}^{n} F(a_i) - F(a_{i-1})$$

$$= \Big(\sum_{i \in A} + \sum_{i \in B}\Big)\big(F(a_i) - F(a_{i-1})\big) \tag{4}$$

where $A = \{i;\ \cancel{x}_i\, a_i \neq c\}$ and $B = \{i;\ \cancel{x}_i = c\}$. N/b x_i

$$\text{Hence} \qquad F(v) - F(u) > -\varepsilon \sum_{i \in A} (a_i - a_{i-1}) - \varepsilon \tag{5}$$

$$> -\varepsilon(b - a) - \varepsilon$$

which, since $\varepsilon > 0$ is arbitrary, gives the result.

The result can easily be extended as follows. (Let n.e. stand for nearly everywhere, that is, except on a countable set.)

Lemma 2'. Let $F: I_o \to R$ be such that

(a) $\underline{D}F \geq 0$ n.e.

(b) $\forall x \in I_o \quad \limsup_{h \to 0+} F(x - h) \leq F(x) \leq \liminf_{h \to 0+} F(x + h)$; then F is increasing.

Proof: If c_n, $n = 1, 2 \ldots$ denotes the countable exceptional set in (a), then if $x \neq c_n$, $n = 1, 2 \ldots$ $\mathcal{I}_x$ is given by (2), while if $x = c_n$ $\mathcal{I}_{c_n}$ is given as in (2) except that $-\varepsilon$ is replaced by $\frac{-\varepsilon}{2^n}$, then the proof proceeds as above.

4.3 A different kind of generalisation of Lemma 1 is given by:

Lemma 3. If $F: I_o \to R$ is such that $\underline{D}F(x) \geq 0$, $x \neq c$ and $\underline{D}F(c) > -\infty$ then F is increasing.

Proof: Proceed as in the proof of lemma 2 up to the definition of $\mathcal{S}_c$. Since $\underline{D}F(c) > -\infty$, there is an $n \in N$ such that $\underline{D}F(c) > -n$. Define

$$\mathcal{S}_c = \{[u, v];\ a \leq u \leq \overset{c}{\cancel{x}} \leq v \leq b,\ v - u < \frac{\varepsilon}{n} \text{ and } F(v) - F(u) > -n(v - u)\}.$$

Then, as in lemma 2, $\tilde{\mathcal{S}}$ is an $\mathcal{O}$-full cover. Let π be as in (3) when $F(v) - F(u)$ is given as in (4). Hence

$$\begin{aligned} F(v) - F(u) &> -\varepsilon \sum_{i \in A} (a_i - a_{i-1}) - n \sum_{i \in B} (a_i - a_{i-1}) \qquad (6) \\ &\geq -\varepsilon(b - a) - \varepsilon \end{aligned}$$

which completes the proof.

Because of the significant difference between (5) and (6) the extension of lemma 3 to lemma 3' is very different from that of lemma 2 to lemma 2'.

Lemma 3'. Let $F: I_o \to R$ be such thta (a) $\underline{D}F \geq 0$ a.e.

(b) $\underline{D}F > -\infty$,

then F is increasing.

Proof: Define E by

$$E = \{x;\ \underline{D}F(x) < 0\}, \tag{7}$$

Then $|E| = 0$. Given $\varepsilon > 0$ if $x \notin E$ we again define $\mathcal{J}_x$ as in (2).

Now let $n \in N$, and (a) define $E_n = \{x;\ -n + 1 > \underline{D}F(x) > -n\}$ and (b) cover E by an open set G_n, $|G_n| < \dfrac{\varepsilon}{n2^n}$. If $x \in E_n$ then let

$$\mathcal{J}_x = \{[u, v];\ a < u < x < v < b,\ [u, v] \subset G_n,\ \text{and } F(v) - F(u) > -n(v-u)\}.$$

Clearly, given the hypotheses $\tilde{\mathcal{J}}$ is an $\mathcal{O}$-full cover and if π is as in (3),

$$F(v) - F(u) = \sum_{i \in A} F(a_i) - F(a_{i-1}) + \sum_{n=1}^{\infty} \sum_{i \in B_n} F(a_i) - F(a_{i-1})$$

where

$$A = \{i;\ x_i \in E\},\quad B_n = \{i;\ x_i \in E_n\},\ n \in N.$$

Then

$$F(v) - F(u) > -\varepsilon \sum_{i \in A} (a_i - a_{i-1}) - \sum_{n=1}^{\infty} n \sum_{i \in B_n} (a_i - a_{i-1})$$

$$\geq - \varepsilon (b - a) - \sum_{n=1}^{\infty} n \frac{\varepsilon}{n2^n} = -\varepsilon(b - a) - \varepsilon,$$

and the lemma is proved.

4.4. Since the definition of an $\mathcal{O}$-cover is a local one, the arguments used to prove lemmas 2' and 3' can be combined to prove

Theorem 1: If $F: I_o \to R$ is such that (a) $\underline{D}F \geq 0$ a.e., (b) $\underline{D}F > -\infty$ n.e., (c) $\limsup_{h\to 0+} F(x - h) \leq F(x) \leq \liminf_{h\to 0+} F(x + h)$ for $x \in I_o$, then F is increasing.

4.5 Since the use of (ρ, λ)-$\mathcal{D}$ full covers, $\rho + \lambda \geq 1$, makes no difference to the above arguments, various non-trivial generalisations of Theorem 1 are immediate. For instance,

Theorem 1': If $F: I_o \to R$ is such that

(a) ap - $\underline{D}F \geq 0$ a.e.

(b) ap - $\underline{D}F > -\infty$ n.e.,

(c) ap - $\limsup_{h\to 0+} F(x - h) \leq F(x) \leq$ ap - $\liminf_{h\to 0+} F(x + h)$, $x \in I_o$,

then F is increasing.

A similar result using preponderant lower derivatives in (a) and (b), and preponderant limits on (c), or even just (ρ, λ)-limits and derivatives, $\rho + \lambda \geq 1$, is readily written down.

4.6 The various generalisations of lemma 1 can be considered as being conditions holding on an exceptional set that make the second term on the right hand side of (4) small, or more precisely not very negative. There is another way of doing this that leads to another type of monotonicity theorem.

Lemma 4. Let $F: I_o \to R$ be $\underline{AC}$ and such that $\underline{D}F \geq 0$ a.e., then F is increasing.

Proof: Given $\varepsilon > 0$, let E be as in (7) and if $x \notin E$ define $\mathcal{J}_x$ as in (2).

Since F is $\underline{AC}$, $\exists \delta > 0$ such tht $\forall\, a_1 < b_1 \leq a_2 \dots < b_n$, with $\sum_{k=1}^{n} (b_k - a_k) < \delta$ we have $\sum_{k=1}^{n} F(b_k) - F(a_k) > -\varepsilon$. Let $I_n \in \mathcal{J}_o$ $n = 1, 2 \dots,$ $E \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \delta$. If $x \in E$, define $\mathcal{J}_x$ as in (0).

$\tilde{\mathcal{J}}$ is an $\mathcal{O}$-full cover. Let π be an $\tilde{\mathcal{J}}$-partition as in (3): then if $A = \{i;\ x_i \quad E\}$ and $B = \{i;\ x_i \in E\}$,

$$F(v) - F(u) = \Big(\sum_{i\varepsilon A} + \sum_{i\varepsilon B} \Big)\big(F(a_i) - F(a_{i-1})\big)$$

$$> -\varepsilon \sum_{i\varepsilon A} (a_i - a_{i-1}) - \varepsilon$$

$$> - \varepsilon(b - \cancel{1}^{a}) - \varepsilon$$

and the lemma is proved.

Lemma 4': Let $F: I_o \to R$ be such that F is continuous, $\underline{D}F(x) \geq 0$, $x \in E$,

$|E| = 0$ and F is $\underline{AC}$ on E, then F is increasing.

Proof: First note that since F is continuous, then F is actually $\underline{AC}$ on $\overline{E}$; (see for instance Saks [8, p.22].)

Proceed exactly as in Lemma 4 and consider the $[a_{i-1}, a_i]$, $i \in B$. Then $x_i \in E$, $a_{i-1} < x_i < a_i$. Put $\alpha_{i-1} = \inf \overline{E} \cap [a_{i-1}, a_i]$, $\alpha_i = \sup \overline{E}\ [a_{i-1}, a_i]$.

By lemma 2' F is increasing an $[a_{i-1}, \alpha_{i-1}]$. Hence

$$\sum_{i \in B} F(a_i) - F(a_{i-1}) \geq \sum_{i \in B} F(\alpha_{i-1}) - F(\alpha_i) > - \varepsilon,$$

as in lemma 4. This is sufficient to prove the lemma.

Theorem 2: If $F: I_o \to R$ is continuous and $\underline{ACG}$ and if $\underline{D}F \geq 0$ a.e. then F is increasing.

Proof: $I_o = \bigcup_{n=1}^{\infty} A_n$ and in each A_n F is $\underline{AC}$. There is no loss in generality in assuming $A_n \cap A_m = \phi$, $n \neq m$. Let $E_n = E \cap A_n$, $n = 1, 2, \ldots$, where E is given by (7).

Since F is AC on $\overline{E}_n$, $\exists\, \delta_n$ such that $\forall\, a_1 < b_1 < a_2 < \ldots < b_m$, points of E_n with $\sum_{k=1}^{m} (b_k - a_k) < \delta_n$, we have $\sum_{k=1}^{m} (F(b_k) - F(a_k)) > \frac{-\varepsilon}{2^n}$.

Let $I_m \in \mathcal{J}_o$, $m = 1, 2, \ldots$, $E_n \bigcup_{m=1}^{\infty} I_m$ and $\sum_{m=1}^{\infty} |I_m| < \delta_n$.

If $x \in E_n$, define $\mathcal{J}_x$ as in (0).

Proceeding as in lemma 4 the second term, summing over $i \in B$, becomes

$$\sum_{n} \sum_{i \in B_n} \left(F(a_i) - F(a_{i-1})\right), \quad B_n = \{i;\ x_i \in E_n\}$$

and the argument of lemma 4' completes the proof.

4.7 The remarks of 4.5 apply here except that unless F is continuous we cannot now say in the proof of Theorem 2 that F is AC on $\overline{E}_n$ unless we assume each A_n is closed. This can be done by replacing ACG by [ACG].

Theorem 2': If $F: I_o \to R$ is approximately continuous and [ACG] and if ap - $\underline{D}F \geq 0$ a.e. then F is increasing.

A similar result using preponderant lower derivatives and preponderant continuity, or even the more general (ρ, λ)-concepts with $\rho + \lambda > 1$, is easily written down.

5. It is well known that if $F: I_o \to R$ is of bounded variation, then F' exists a.e., and is L-integrable, but that F is not necessarily an L-primitive of F'. If however F' exists everywhere, then F is in fact absolutely continuous, and F is therefore an L-primitive of F. This result, due to Lebesgue, can be found in Hobson, [5, p.549]. A simple way of seeing this is to note that in this case F' is Perron integrable and so F is ACG* and continuous. An L-primitive of F' is also ACG* and continuous (being of course absolutely continuous). Two continuous ACG* functions having the same derivative, F', a.e. differ by a constant and so F is an L-primitive of F'.

The purpose of this section is to give a simple direct proof of this result.

Theorem 3: If $F: I_o \to R$ is such that F' is finite and L-integrable, then F is absolutely continuous.

Proof: Given $\varepsilon > 0$ choose $c_n \in R$, $n \in Z$, such that:

(i) $0 < c_{n+1} - c_n < \varepsilon$, $n \in Z$,

(ii) $\lim_{n\to-\infty} c_n = -\lim_{n\to\infty} c_n = -\infty$,

(iii) if $E_n = \{x;\ c_n < F'(x) \leq c_{n+1}\}$, then

$$\left|\int_a^b |F'| - \sum_{n\varepsilon Z} |c_n|\ |E_n|\right| < \varepsilon.$$

Now choose $\varepsilon_n > 0$, $n \varepsilon Z$, such that $\sum_{n\varepsilon Z} \varepsilon_n |c_n| < \varepsilon$, and $I_n^k \varepsilon \mathcal{J}_o$, $n \varepsilon Z$, $k = 1, 2, \ldots$ such that $E_n \subset \bigcup_k I_n^k$, and $\sum_k |I_n^k| < |E_n| + \varepsilon_n$.

Consider the family

$$\mathcal{J} = \{[u, v];\ a \leq u < v \leq b,\ |c_n| - \varepsilon < \left|\frac{F(v) - F(u)}{v - u}\right| < |c_n| + \varepsilon,$$

$$[u, v] \subset I_n^k,\ n \varepsilon Z,\ k = 1, 2 \ldots\}.$$

Then $\mathcal{J}$ is an $\mathcal{O}$-cover and suppose π, as in (3), is an $\mathcal{J}$-partition of $[a, b]$. Let $A_n^k = \{i;\ x_i \varepsilon I_n^k\}$, $n \varepsilon Z$, $k = 1, 2, \ldots$, then

$$\begin{aligned}
|F(b) - F(a)| &\leq \sum_{n,k} \sum_{i\varepsilon A_n^k} |F(a_i) - F(a_{i-1})| \\
&\leq \sum_{n,k} \sum_{i\varepsilon A_n^k} (|c_n| + \varepsilon)(a_i - a_{i-1}) \\
&\leq \sum \sum_{A\ k} |c_n|\ |I_n^k| + \varepsilon(b - a)
\end{aligned}$$

$$< \int_a^b |F'| + \varepsilon(2 + b - a)$$

Since in this argument [a, b] can be replaced by any [c, d] $\subset$ [a, b] this suffices to prove the theorem.

Corollary 3': If F is of bounded variation and F' exists everywhere, then $F(x) - F(a) = L - \int_a^x F'$.

The proof of the above theorem and corollary remains essentially the same if we assume F'_{ap} exists, or even that F'_{pr} exists.

6. As we have seen the proofs of Theorems 1-3 are both simple and easily extended to any general derivative which defines a partitioning cover, such as the approximate or preponderant derivatives. However similar results for unilateral derivatives (Theorems 1 and 2), or upper and lower derivatives, unilateral or bilateral, (Theorem 3) cannot be obtained by our method as the associated full covers are not partitioning; (see Thomson [9, 10].

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