Real Analysis Exchange Vol. 9 (1983-24

A.M. Bruckner[1], Mathematics Department, University of California, Santa Barbara, California 93106.

B.S. Thomson[2], Mathematics Department, Simon Fraser University, B.C., Canada V5A 1S6.

POROSITY ESTIMATES FOR THE DINI DERIVATIVES[3]

§1. Introduction. The notion of set porosity has, in recent years, found a renewed application to certain problems in the differentiation theory of real functions. As a local concept it arose originally in the work of Denjoy some sixty years ago but attracted little notice until the introduction of the notion of σ-porosity by Dolženko in 1967. Since then these ideas have been applied in a number of investigations.

Globally the concept provides a class of exceptional sets (the σ-porous sets, the σ-(ψ)-porous sets) that permits a refinement in many cases of the notion of a first category set. Locally, porosity can be used to provide certain insights into the differentiability behaviour of real functions.

A.M.S.(MOS) Subject Classification: Primary 26A24 ; Secondary 26A21. Key words and phrases: Dini derivatives, porosity, approximate derivatives.

[1]The research of this author was supported in part by a grant from the National Sciences Foundation.

[2]The research of this author was supported in part by a grant from the National Sciences and Engineering Research Council of Canada.

[3]This work was prepared during the Special Year in Real Analysis (1983-84) at U.C.S.B.

In this article we explore some of the applications of porosity in differentiation theory. Most of the results follow directly from the elementary porosity estimates that are made in section 2. To describe the results in the most immediate language we have chosen to introduce the notion of a "porosity derivative", which is a generalized derivation much like approximate derivation but using porosity requirements rather than density requirements. Each of the results presented attempts to show how it is that some information about the porosity derivatives of a function may give rise to information about the ordinary derivates.

The language is rather technical and so it would seem appropriate to illustrate the ideas with some preliminary comments. Suppose that a continuous function f and a positive number M are given, and that the sets

$$E_x = \left[y : \left| \frac{f(y) - f(x)}{y - x} \right| \leqq M,\ y \neq x \right]$$

are constructed at each point x. If each of these sets E_x contains a sequence converging to x, then one can assert, at most, that the function f has a derived number at each point x whose value is in the interval $[-M,M]$.

This does not give much information about the function f. Presumably if more information about the sets E_x is available, then greater knowledge of the behaviour of the function may be deduced. For example at the extreme case if each set E_x is a (deleted) neighborhood of x, then it is easy to prove that the function f satisfies a Lipschitz condition

$$|f(x) - f(y)| \leqq M|x - y| .$$

If however one is told less, say that each set E_x has upper density 1 at x, then the function may still be quite ill-behaved; for example this may occur at almost every point x and yet f may fail to have an approximate derivative at any point.

There is some information which, at first sight, may appear marginal and yet from which some reasonable behaviour of the function f may be deduced. If each set E_x contains on one side or the other at x a sequence of points $\{x_n\}$ that converges to x but not too quickly, i.e. so that

$$\liminf_{n \to \infty} \left| \frac{x_{n+1} - x}{x_n - x} \right| > 0,$$

then there must be a dense open set G so that f is differentiable almost everywhere on G. In the language of set porosity this sequence is required to have porosity less than 1; in the language of porosity derivatives the function f is required to have a certain type of derivative that assumes its values in the interval $[-M,M]$. (See Theorem (5.3) ff.)

It is concerns of this nature that arise in the article. In each case information about the derivates of a function is obtained from some porosity computation, which may always be reduced to assertions about derived numbers taken relative to certain slowly converging sequences.

§2. Porosity Computations. We give the necessary definitions of the porosity notions that we require.

(2.1) DEFINITION. Let E be any set of real numbers and let $a < b$. By $\lambda(E,a,b)$ we denote the length of the largest subinterval of (a,b) that is complementary to the set E. Then the two porosities (left and right) of E at a point x are defined as the extreme limits

$$p^+(E;x) = \limsup_{h\to 0+} \frac{\lambda(E,x,x+h)}{h} ,$$

and

$$p^-(E;x) = \limsup_{h\to 0+} \frac{\lambda(E,x-h,x)}{h} .$$

Loosely the porosity measures the relative sizes of the gaps in E. A natural generalization of this is to measure this relative size in some other appropriate fashion. If $0 < \alpha < 1$, then we may define in a similar way the notion of the (x^α) - porosity by writing

$$p_\alpha^+(E;x) = \limsup_{h\to 0+} \frac{[\lambda(E,x,x+h)]^\alpha}{h} ,$$

and

$$p_\alpha^-(E;x) = \limsup_{h\to 0+} \frac{[\lambda(E,x-h,x)]^\alpha}{h} .$$

More generally still if ψ is some nonnegative real function, we may write

$$p_\psi^+(E;x) = \limsup_{h\to 0+} \frac{\psi(\lambda(E,x,x+h))}{h} ,$$

and

$$p_\psi^-(E;x) = \limsup_{h\to 0+} \frac{\psi(\lambda(E,x-h,x))}{h} .$$

Note that if we permit as a limiting case that the function ψ have $\psi(0) = 0$, $\psi(t) = +\infty$ $(t > 0)$, then a set E will have zero p_ψ^+ (p_ψ^-) porosity at a point x if and only if E is dense on the right (left) at the point x.

In order to provide some substance and insight into the nature of sets having various porosity requirements we consider some examples. A sequence $\{h_n\}$ converging to zero is said to have a porosity p in one of these senses (right porosity, left porosity, right (x^α)-porosity, etc.) if the set of points

$$\{ h_1, h_2, h_3, h_4, \dots \}$$

has that porosity at the point 0. If $\{ h_n \}$ is a descending sequence with $\lim h_n = 0$ and if

$$\liminf_{n\to\infty} \frac{h_{n+1}}{h_n} = r ,$$

then the sequence has right porosity $1 - r$. Thus slowly converging sequences have zero porosity; similarly a sequence that has porosity 1 must converge to zero quite quickly. In the case of (x^α)-porosity the number s $(0 \leqq s \leqq +\infty)$

$$s = \limsup_{n\to\infty} \frac{(h_n - h_{n+1})^\alpha}{h_n}$$

is the right (x^α)-porosity of the sequence $\{h_n\}$. If a sequence has right (x^α)-porosity finite $(s < +\infty)$ for some $0 < \alpha < 1$, then the sequence has right porosity 0 in the ordinary sense, and also too in the (x^β)-sense for any $\alpha < \beta \leqq 1$. Thus again while for zero porosity the sequence must be slowly converging to zero, for the (x^α)-porosity to be finite it must be even more slowly converging to zero. These concepts allow a precise language for "slowly converging to zero" together with a tight interrelationship between these notions and various estimates for the Dini derivatives of various classes of functions.

We provide an example to show how one may generate sequences that exhibit certain porosity behaviour.

(2.2) Example. Suppose that ψ is a continuous strictly increasing function on $[0,+\infty)$ such that

$$\psi(t) = o(t^{1/k+1}).$$

We show how a sequence of numbers $\{x_n\}$ may be constructed that has zero (ψ)-porosity on the right.

Let $\alpha(x) = x - x^{k+1}$. For any $x \in (0,1)$ define the sequence $\{x_n\}$ by writing inductively

$$x_1 = \alpha(x) \ , \quad x_2 = \alpha(x_1) \ , \ \dots \ x_j = \alpha(x_{j-1}).$$

We claim that this sequence must have zero (ψ)-porosity on the right. The sequence $\{x_n\}$ is decreasing to zero and so the porosity computation requires that we establish the limit

$$\lim_{t\to 0+} \frac{\psi(t - \alpha(t))}{t} = 0.$$

But

$$\frac{\psi(t - \alpha(t))}{t} = \frac{\psi(t^{k+1})}{t}$$

and for t sufficiently close to zero

$$\psi(t) < t^{1/(k+\frac{1}{2})}.$$

This gives for small t,

$$\frac{\psi(t - \alpha(t))}{t} = \frac{\psi(t^{k+1})}{t}$$

$$< \quad t^{(k+1)/(k+\frac{1}{2})-1} = t^{1/(2k+1)}$$

and the limit is established as required.

In order to provide some feeling for the nature of such sequences we carry through the necessary computations in order to provide a sequence of numbers which has zero $(x^{\frac{1}{2}})$-porosity. The sequence $\{x_n\}$ is computed as above using the function $\alpha(x) = x - x^3$.

n	x_n	x_n	x_n	x_n
1	.100000000000	.010000000000	.001000000000	.000100000000
2	.099000000069	.009999000000	.000999999000	.000099999999
3	.098029701062	.009998000300	.000999998000	.000099999998
4	.097087653028	.009997000900	.000999997000	.000099999997
5	.096172503685	.009996001800	.000999996000	.000099999996

Observe that even for $x_1 = 0.0001$, the sequence of numbers $\{x_n\}$ is already very slowly converging.

The basic computation from which all of our results follows is a simple estimate on the porosity of a set that arises in the comparison of the Dini derivatives. It first appears explicitly as a porosity computation in the article of Evans and Humke [6] but considerations of this type can be found in a number of theorems. For example the proof of Mišík's theorem in Bruckner [1,p.154] uses such an estimate and the sequential derivation problems in Shukla [13] and in Laczkovich and Petruska [9] require some such calculations. Doubtless similar technical details can be found in much earlier reports.

(2.3) POROSITY LEMMA. Let f be a monotonic nondecreasing function such that at a point x_0 one has $\underline{D}^+f(x_0) \leqq r < s$. Then the set of points

$$Y = \left[y : \frac{f(y) - f(x_0)}{y - x_0} \geqq s \right]$$

has right porosity at x_0 at least $1 - \frac{r}{s}$. Similarly if $\overline{D}^+f(x_0) \geqq s > r$, then the set of points

$$Y = \left[y : \frac{f(y) - f(x_0)}{y - x_0} \leqq r \right]$$

has right porosity at x_0 at least $1 - \frac{r}{s}$.

PROOF. The computational details appear in full in Thomson [15,p.418-419].

We give an example to show that the estimate in the lemma is sharp.

(2.4) Example. For any number $0 < p < 1$ take the sequence

$$h_n = (1-p)^{2n}$$

and define the function f by setting

$$f(x) = (1-p)h_{n-1} \qquad (h_n < x \leqq h_{n-1})$$

and $f(x) = 0$ otherwise. Then one checks easily that $\underline{D}^+f(0) = 1 - p$ and the set Y,

$$Y = \left[y : \frac{f(y) - f(0)}{y} \geqq 1 \right]$$

is complementary to the intervals

$$((1-p)h_{n-1}, h_{n-1}].$$

Thus Y has porosity p on the right at 0, which is just the estimate that the lemma provides.

This porosity lemma generalizes to much broader classes of functions. For ordinary Lipschitz functions (i.e. functions f that satisfy everywhere an inequality of the form

$$|f(x) - f(y)| \leqq M|x - y|$$

for some number $M > 0$) a generalization is immediately available. We observe that for such a function f the functions $f(x) \pm Mx$ are necessarily monotonic. This leads to the following lemma. (Alternatively the computations can be done directly (see Thomson [15,pp.419,420])).

(2.5) LEMMA. Let f satisfy a Lipschitz condition

$$|f(x) - f(y)| \leqq M|x - y|$$

for some $M > 0$. If at a point x_0 one has $\underline{D}^+ f(x_0) \leqq r < s$, then the set of points

$$Y = \left[y : \frac{f(y) - f(x_0)}{y - x_0} \geqq s \right]$$

has right porosity at x_0 at least $1 - \frac{M + r}{M + s}$. Similarly if $\overline{D}^+ f(x_0) \geqq s > r$, then the set of points

$$Y = \left[y : \frac{f(y) - f(x_0)}{y - x_0} \leqq r \right]$$

has right porosity at x_0 at least $1 - \frac{M + r}{M + s}$.

Even more generally the porosity lemma permits a generalization to the class of continuous functions f that satisfy an inequality of the form

$$|f(x) - f(y)| \leqq \psi(|x - y|)$$

where ψ is a given modulus of continuity. That is to say ψ is defined for all nonnegative reals, is increasing,

$$\lim_{x \to 0+} \psi(x) = \psi(0) = 0$$

and for all real numbers x and y if $x - y \leqq 1$, then

$$|f(x) - f(y)| \leqq \psi(|x - y|) .$$

The class of continuous functions that permit such an inequality shall be denoted as $C(\psi)$. Of course the most interesting special case occurs with $\psi(x) = Mx^{\alpha}$ for $0 < \alpha \leqq 1$, and for M a positive real constant.

Note that the extreme case with $\psi(t) = +\infty$ $(t > 0)$ and $\psi(0) = 0$ is not permitted by the assertion of the lemma, but that the lemma is nonetheless true for such a function ψ since in that case positive (ψ)-porosity of a set A at a point is equivalent to the nondenseness of the set A at that point.

(2.6) LEMMA. Let f be a continuous function that satisfies an inequality

$$f(x) - f(y) \leqq \psi(|x - y|)$$

for a modulus of continuity ψ that is defined and continuous on $[0,+\infty)$, with $\psi(0) = 0$ and with $\psi_+'(0) = +\infty$. Suppose that at a point x_0 one has $\underline{D}^+f(x_0) \leqq r < s$. Then the set of points

$$Y = \left[y : \frac{f(y) - f(x_0)}{y - x_0} \geqq s \right]$$

has right (ψ)-porosity at x_0 at least $s - r$. Similarly if $\overline{D}^+f(x_0) \geqq s > r$, then the set of points

$$Y = \left[y : \frac{f(y) - f(x_0)}{y - x_0} \leqq r \right]$$

has right (ψ)-porosity at x_0 at least $s - r$.

PROOF. We prove the first statement of the lemma. Since $\underline{D}^+f(x_0) < r$, we may select a sequence of positive numbers $\{h_n\}$ descending to zero so that

$$\frac{f(x_0 + h_n) - f(x_0)}{h_n} < r .$$

We will show how to choose numbers $\{\theta_k\}$, $0 < \theta_k < 1$, and a subsequence $\{h_{n_k}\}$ in such a way that $\theta_k \to 1$,

$$\frac{1}{\theta_k}\left[\frac{\psi(h_{n_k}(1 - \theta_k))}{h_{n_k}} + r\right] = s$$

and

$$\lim_{k\to\infty} \frac{\psi((1 - \theta_k)h_{n_k})}{h_{n_k}} = s - r .$$

Let us suppose for the moment that these numbers can be so chosen. If so, then we can obtain the porosity estimate promised in the statement of the lemma. Consider the set of points

$$Y = \left[y : \frac{f(y) - f(x_0)}{y - x_0} \geqq s \right]$$

and the sequence of intervals

$$\{(x_0 + \theta_k h_{n_k}, x_0 + h_{n_k})\} .$$

We observe that each of these intervals is necessarily disjoint from Y since if there is given a point y,

$$x_0 + \theta_k h_{n_k} < y < x_0 + h_{n_k} ,$$

we must have using the ψ-inequality on f that

$$\frac{f(y) - f(x_0)}{y - x_0}$$

$$= \frac{f(y) - f(x_0 + h_{n_k})}{y - x_0} + \frac{f(x_0 + h_{n_k}) - f(x_0)}{h_{n_k}} \frac{h_{n_k}}{y-x_0}$$

$$< \frac{\psi(y - x_0 - h_{n_k})}{y - x_0} + \frac{r}{\theta_k}$$

$$< \frac{1}{\theta_k}\left[\frac{\Psi((1 - \theta_k)h_{nk})}{h_{n_k}} + r\right] = s .$$

As each interval $(x_0 + \theta_k h_{n_k}, x_0 + h_{n_k})$ is now seen to be disjoint from the set Y, we compute that the (ψ)-porosity of Y on the right must exceed the number

$$\limsup_{k\to\infty} \frac{\psi((1-\theta_k)h_{n_k})}{h_{n_k}} .$$

But, from the way in which these sequences have been chosen, this limit is $s - r$ which is exactly the porosity which we were required to obtain.

Thus it remains for us to prove that these sequences may be selected in the way that we have stated. Consider in the (ξ,y) plane for $0 \leqq \xi \leqq 1$ the straight line

$$y = s\xi - r$$

and for any $h > 0$, the curve

$$y = \frac{1}{h}\,\psi((1-\xi)h) .$$

For fixed $\xi < 1$,

$$\lim_{h\to 0+} \frac{\psi((1-\xi)h)}{h} = (1-\xi)\psi_+'(0) = +\infty.$$

Note that the line passes through the point $(1,s-r)$, and the curve passes through the point $(1,0)$. This allows us to select points $\xi_1,\xi_2,\xi_3,\ \ldots$ and indices $n_1,n_2,n_3,\ \ldots$ inductively so that $n_1 = 1$, and

$$s\xi_1 - r > \frac{\psi((1-\xi_1)h_1)}{h_1} ,$$

$$s\xi_1 - r < \frac{\psi((1-\xi_1)h_{n_2})}{h_{n_2}} ,$$

$$\xi_2 > 1 - \tfrac{1}{2},$$

$$s\xi_2 - r > \frac{\psi((1-\xi_2)h_{n_2})}{h_{n_2}} ,$$

and so on in this fashion so that $\{\xi_j\}$ and $\{h_{n_j}\}$ are obtained and satisfy the inequalities

$$\xi_j > 1 - \frac{1}{j}$$

$$s\xi_j - r < \frac{\psi((1-\xi_j)h_{n_{j+1}})}{h_{n_{j+1}}} .$$

$$s\xi_{j+1} - r > \frac{\psi((1-\xi_{j+1})h_{n_{j+1}})}{h_{n_{j+1}}} .$$

Since ψ is continuous, we may choose numbers $\{\theta_{j+1}\}$ from the intervals (ξ_j,ξ_{j+1}) so that

$$s\theta_{j+1} - r = \frac{\psi((1-\theta_{j+1})h_{n_{j+1}})}{h_{n_{j+1}}}$$

and it is clear that these sequences are precisely what was required in order to complete the proof.

§3. Estimates for the Dini derivatives. Our estimates require a generalized version of the Dini derivative. We shall consider this a porosity version of the usual Dini derivative and we define it in a way that should be familiar to the reader. (For example compare with the definition of the approximate Dini derivatives in Saks [12,p.219]).

(3.1) DEFINITION. Let f be a real function and let $0 \leqq p < 1$. Then at any point x we define the four porosity Dini derivatives, with index p, as

$$\overline{PD}_p^{+}f(x) = \inf\{ y: \left[t: \frac{f(t)-f(x)}{t-x} < y\right] \text{ has right porosity} \leqq p \text{ at } x\},$$

$$\underline{PD}_p^{+}f(x) = \sup\{ y: \left[t: \frac{f(t)-f(x)}{t-x} > y\right] \text{ has right porosity} \leqq p \text{ at } x\},$$

$$\overline{PD}_p^{-}f(x) = \inf\{ y: \left[t: \frac{f(t)-f(x)}{t-x} < y\right] \text{ has left porosity} \leqq p \text{ at } x\},$$

and

$$\underline{PD}_p^{-}f(x) = \inf\{ y: \left[t: \frac{f(t)-f(x)}{t-x} > y\right] \text{ has left porosity} \leqq p \text{ at } x\}.$$

Similarly using (x^{α})-porosity and any number $0 \leqq t < +\infty$ we may define the versions $\overline{PD}_{\alpha,t}^{+}f(x)$, $\overline{PD}_{\alpha,t}^{-}f(x)$, $\underline{PD}_{\alpha,t}^{+}f(x)$, and $\underline{PD}_{\alpha,t}^{-}f(x)$; more generally still using an appropriate function ψ and a number $0 \leqq t < +\infty$ we may define the versions $\overline{PD}_{\psi,t}^{+}f(x)$, $\overline{PD}_{\psi,t}^{-}f(x)$, $\underline{PD}_{\psi,t}^{+}f(x)$, and $\underline{PD}_{\psi,t}^{-}f(x)$ using ψ-porosity.

For any indices $0 \leqq p \leqq q \leqq 1$ one must have the inequalities

$$\underline{D}^{+}f(x) \leqq \overline{PD}_q^{+}f(x) \leqq \overline{PD}_p^{+}f(x) \leqq \overline{D}^{+}f(x)$$

and

$$\underline{D}^{+}f(x) \leqq \underline{PD}_p^{+}f(x) \leqq \underline{PD}_q^{+}f(x) \leqq \overline{D}^{+}f(x) .$$

Note, however that there need be no relation between the upper and lower porosity derivatives themselves; indeed the lower may exceed the upper.

The basic estimates on the Dini derivatives that we require are contained in the next few assertions. Each is just an easy consequence of the porosity computations of the preceding section, but expressed in terms of a porosity version of the Dini derivatives.

(3.2) LEMMA. Let f be monotonic nondecreasing on a neighborhood of a point x_0 and let p be a number, $0 \leqq p < 1$. Then at x_0 one has the inequality

$$(1-p)\underline{PD}_p{}^+f(x_0) \leqq \underline{D}^+f(x_0) \leqq \overline{D}^+f(x_0) \leqq \frac{1}{1-p}\,\overline{PD}_p{}^+f(x_0) \ .$$

PROOF. Suppose that $\underline{D}^+f(x_0) < r$. Then by the porosity lemma (2.3) the set of points

$$Y = \quad y : \frac{f(y) - f(x_0)}{y - x} \geqq s > r$$

has right porosity at x_0, exceeding $1 - \frac{r}{s}$. If $\underline{PD}_p{}^+f(x_0) > s$, then this set Y needs to have porosity less than p. This then requires that

$$1 - \frac{r}{s} < p$$

or equivalently

$$(1 - p)\ s < r.$$

The first inequality in the theorem now follows; the last can be obtained in a similar fashion.

The remaining lemmas are similarly proved using the other versions of the basic porosity computations. We omit the details.

(3.3) LEMMA. Let f satisfy a Lipschitz condition,

$$|f(x) - f(y)| \leqq M|x - y|$$

and let p be a number, $0 \leqq p < 1$. Then at any point x_0 one has the inequalities

$$-Mp + (1-p)\underline{PD}_p^+ f(x_0) \leqq \underline{D}^+ f(x_0) \leqq \overline{D}^+ f(x_0) \leqq \frac{1}{1-p}\overline{PD}_p^+ f(x_0) + Mp.$$

(3.4) LEMMA. Let f satisfy an inequality of the form

$$|f(x) - f(y)| \leqq M|x - y|^\alpha \quad (|x - y| \leqq 1)$$

for some numbers $M > 0$ and $0 < \alpha < 1$, and let t be a number, $0 \leqq t < +\infty$. Then at any point x_0 one has the inequalities

$$-Mt + \underline{PD}_{\alpha,t}^+ f(x_0) \leqq \underline{D}^+ f(x_0) \leqq \overline{D}^+ f(x_0) \leqq \overline{PD}_{\alpha,t}^+ f(x_0) + Mt.$$

(3.5) LEMMA. Let f be a continuous function that satisfies an inequality of the form

$$|f(x) - f(y)| \leqq \psi(|x - y|) \quad (|x - y| \leqq 1)$$

where ψ is a continuous increasing function on $[0,+\infty)$ for which $\psi(0) = 0$ and for which $\psi_+'(0) = +\infty$. Suppose that t is a number, $0 \leqq t < +\infty$. Then at any point x_0 one has the inequalities

$$-t + \underline{PD}_{\psi,t}^+ f(x_0) \leqq \underline{D}^+ f(x_0) \leqq D^+ f(x_0) \leqq \overline{PD}_{\psi,t}^+ f(x_0) + t.$$

§4. Basic results. Each of the results in this section is a direct and easy consequence of the preceding estimates. It is a remarkable fact that so many observations can be made to rely directly on these elementary computations.

(4.1) THEOREM. Let f be monotonic or Lipschitz. Then at every point the four Dini derivatives and the four porosity zero Dini derivatives agree:

$$\overline{PD}_0^{\,+}f(x_0) = \overline{D}^+f(x_0), \qquad \underline{PD}_0^{\,+}f(x_0) = \underline{D}f(x_0),$$

and

$$\overline{PD}_0^{\,-}f(x_0) = \overline{D}^-f(x_0), \quad \underline{PD}_0^{\,-}f(x_0) = \underline{D}^-f(x_0).$$

PROOF. With $p = 0$ in lemma (3.2) and (3.3) this is immediate.

(4.2) THEOREM. Let f be monotonic nondecreasing at a point x_0 and suppose that $0 \leqq p < 1$. If $\underline{PD}_p^{\,+}f(x_0) = +\infty$, then it must be the case that $f'_+(x_0) = +\infty$; if $\overline{PD}_p^{\,+}f(x_0) = 0$, then it must be the case that $f'_+(x_0) = 0$.

PROOF. This too is an immediate consequence of lemma (3.2).

These results apply immediately to provide some well known estimates for the approximate derivative and approximate derivates of monotonic functions. This gives us a theorem of Khintchine [8] and of Misik [10]. We write $f'_{ap}(x_0)$ for the approximate derivative of a function f at a point x_0, and we write $\underline{D}_{ap}^{\,+}f(x_0)$, $\underline{D}_{ap}^{\,-}f(x_0)$, etc. for the approximate Dini derivatives of f at this point.

(4.3) THEOREM. [Khintchine] Let f be monotonic or Lipschitz. If $f'_{ap}(x_0)$ exists at a point x_0 then f must be differentiable there.

PROOF. This follows directly from the observation that sets having density 1 at a point must have porosity zero at that point. Consequently at any point the inequalities

$$\underline{D}^{+}f(x) \leqq \underline{D}_{ap}^{+}f(x) \leqq \underline{PD}_{0}^{+}f(x)$$

and

$$\overline{D}^{+}f(x) \geqq \overline{D}_{ap}^{+}f(x) \geqq \overline{PD}_{0}^{+}f(x)$$

must hold, along with similar assertions for left hand derivates. The proof is then completed by an application of theorem (4.1).

(4.4) THEOREM. [Mišík] Let f be either a monotonic or a Lipschitz function. Then at any point x_0

$$\overline{D}^{+}f(x_0) = \overline{D}_{ap}^{+}f(x_0), \qquad \underline{D}^{+}f(x_0) = \underline{D}_{ap}^{+}f(x_0),$$

$$\overline{D}^{-}f(x_0) = \overline{D}_{ap}^{-}f(x_0), \qquad \underline{D}^{-}f(x_0) = \underline{D}_{ap}^{-}f(x_0).$$

PROOF. The same observations that were used in the preceding theorem supply the proof.

There are several results that are merely restatements of the fact that a function that is monotonic or Lipschitz and differentiable at a point x_0 relative to a set that is nonporous at x_0 must be differentiable. In each case the proof is obvious and is omitted.

(4.5) THEOREM. Let f be monotonic or Lipschitz and differentiable relative to a set E at each point of E, i.e. at each $x \in E$ the limit

$$\lim_{y \to x,\ y \in E} \frac{f(y) - f(x)}{y - x}$$

exists. Then f is differentiable at each point of E with the possible exception of a porous set.

(4.6) THEOREM. Let f be monotonic or Lipschitz and differentiable relative to each of a sequence of sets $\{E_n\}$, i.e. at each point $x \in E_n$ and for any $n = 1,2,3,\ldots$ the limit

$$\lim_{y \to x,\ y \in E_n} \frac{f(y) - f(x)}{y - x}$$

exists. Then f is differentiable at each point in the union of the sets $\{E_n\}$ with the possible exception of a σ-porous set.

Note in particular that as a consequence of this if such a function f is differentiable relative to each of a sequence of sets that covers the entire real line, then it must be differentiable everywhere except on a σ-porous set. Also if P is a perfect null set that is not σ-porous (because of Zajíček [17] we know that such sets exist) and a monotonic function f is not differentiable at any point of P, then certainly, by (4.6), P may not be expressed as a union of a sequence of sets $\{E_n\}$ so that f is differentiable relative to each member of the sequence.

(4.7) THEOREM [Evans-Humke] Let f be monotonic or Lipschitz. Then at every point x with the possible exception of a σ-porous set the relations

$$\overline{D}^{+}f(x) = \overline{D}^{-}f(x) = \overline{D}f(x)$$

$$\underline{D}^{+}f(x) = \underline{D}^{-}f(x) = \underline{D}f(x)$$

must hold.

PROOF. As in most theorems of this type one considers the set of points

$$X_{rs} = \left[x : \overline{D}^{+}f(x) < r < s < \overline{D}^{-}f(x)\right]$$

for r,s rational. This set is partitioned into a sequence $\{X_{rsn}\}$ in a familiar way (cf. Saks [12,pp.237-238]) so that for pairs of points $x,y \in X_{rsn}$, $x < y$ the inequality

$$\frac{f(y) - f(x)}{y - x} < r$$

holds. Then the porosity lemma applies to show that each set X_{rsn} must have positive porosity at each of its points. As the exceptional set of the theorem may be expressed as a denumerable union of such sets the theorem follows.

Theorem (4.4) generalizes easily and with an identical proof to apply to the class of functions $C(\psi)$ for a fixed modulus of continuity ψ. Again we assume that ψ is a continuous increasing function for which $\psi(0) = 0$ and $\psi_+'(0) = +\infty$.

(4.8) THEOREM. Let f be a function in class $C(\psi)$. Then the relations

$$\overline{D}^+f(x) = \overline{D}^-f(x) = \overline{D}f(x)$$

and

$$\underline{D}^+f(x) = \underline{D}^-f(x) = \underline{D}f(x)$$

hold at every point x with the possible exception of a σ-(ψ)-porous set.

§5. Generalized Young-Evans-Humke theorem. The Evans-Humke theorem as given above together with its generalization to the class $C(\psi)$ in theorem (4.8) really belongs in a hierarchy of theorems ranging from a theorem of W.H.Young [16] that for a continuous function f

the set of points of right and left disagreement

$$\left[x : \overline{D}^+f(x) \neq \overline{D}^-f(x) \quad \text{or} \quad \underline{D}^+f(x) \neq \underline{D}^-f(x)\right]$$

is first category, through the various classes $C(\psi)$ each time improving the exceptional set beyond merely first category to some σ-(ψ)-porous set.

This class of theorems permits yet another type of generalization. In place of discovering a comparison between the right and left ordinary Dini derivatives one can ask how far this extends to thinner derivates. For example Pu,Chen and Pu [11] have checked that this theorem of Young extends to the approximate Dini derivatives and Zajiček [17] has pushed it (and the Evans-Humke theorem) further to accomodate extremely small density derivatives. Possibly the correct version involves just the porosity Dini derivatives. For continuous functions this was proved in Thomson [14], and for monotonic functions it was announced without proof in Thomson [15,p.340]. Here we prove that this theorem is available for any class $C(\psi)$ with an exceptional set that is σ-(ψ)-porous. Note that at the extreme end of the spectrum where $\psi(0) = 0$, $\psi(x) = +\infty$ $(x > 0)$ the exceptional set that is σ-(ψ)-porous is required merely to be first category as is the case in the original theorem of Young.

(5.1) THEOREM. Let f belong to $C(\psi)$, that is to say f is a continuous function that satisfies an inequality of the form

$$|f(x) - f(y)| \leq \psi(|x - y|) \qquad (|x - y| \leq 1)$$

where ψ is a continuous increasing function on $[0,+\infty)$ for which $\psi(0) = 0$ and $\psi_+'(0) = +\infty$. Then at every point x with the possible exception of a set that is σ-(ψ)-porous, and for every $0 \leqq p < 1$,

$$\overline{PD}_p^{\,+}f(x) = \overline{PD}_p^{\,-}f(x) = \bar{D}f(x)$$

and

$$\underline{PD}_p^{\,+}f(x) = \underline{PD}_p^{\,-}f(x) = \underline{D}f(x).$$

PROOF. We shall show that for any fixed $q < p < 1$ and pair of rational numbers r,s the set of points

$$X_{rs} = \left[x : \overline{PD}_q^{\,+}f(x) < r < s < \bar{D}^-f(x)\right]$$

is σ-(ψ)-porous whenever $f \in C(\psi)$. In view of the weaker version of this theorem given earlier (theorem (4.8)) this is enough to obtain a proof.

At each point $x \in X_{rs}$ define the set S_x as

$$S_x = \left[y : \frac{f(y) - f(x)}{y - x} < r\right]$$

and note that the set S_x has right porosity at x no larger than $q < p$. Thus we may choose for each such point x a positive number $\delta(x)$ so that

$$\lambda(S_x, x, x+t) < pt$$

whenever $0 < t < \delta(x)$. The function δ then induces, in a familiar way (cf. Bruckner, O'Malley and Thomson [2]) a partition of X_{rs} into a sequence of sets $\{X_{rsn}\}$ so that each pair of points $x,y \in X_{rsn}$ satisfies

$$|x - y| < \min [\delta(x), \delta(y)].$$

We show that each set X_{rsn} has positive (ψ)-porosity at each of its points. Suppose, in order to obtain a contradiction, that this is not so. Then at some point x_0 in X_{rsn} this set has (ψ)-porosity zero at x_0. Thus we can find a positive number η so that

$$\psi(\lambda(X_{rsn}, x_0 - t, x_0)) < \nu t$$

for $0 < t < \eta$, where ν is taken as follows: firstly we define numbers θ and τ from the interval $(0,1)$ in such a way that

$$\tau < (s - r)(1 - \theta)$$

and

$$\theta > p(1 + \tau).$$

Since $s > r$ and $0 < p < 1$ these are simple linear inequalities that may be solved. Then we take

$$\nu < \frac{\tau}{1 + \tau} .$$

Take any point x_1 in X_{rsn} with $x_1 < x_0$, and $x_1 > x_0 - \eta$; we claim that the inequality

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1} \leqq s$$

must hold at these points. As will be shown later this inequality will provide the (ψ)-porosity estimate on X_{rsn}. Most of the rest of the proof is devoted to proving the inequality.

To verify this inequality we define

$$x_2 = \sup \left[z \in (x_1, x_0) : \frac{f(z) - f(x_1)}{z - x_1} \leq s \right]$$

and we prove that $x_2 = x_0$. As f is continuous this forces the required inequality

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1} \leq s$$

and our claim is proved. We obtain this now by a contradiction; if contrary to this $h = x_0 - x_2 > 0$, then we consider the interval

$$(x_2 - \psi^{-1}(\tau h), x_2)$$

where τ is as chosen above and ψ^{-1} is the inverse function to ψ. We will take it so that $\psi^{-1}(\tau h) < \tau h$ which requires that τ has been chosen sufficiently small.

Using the porosity condition on X_{rsn} and the inequality

$$\frac{\psi[\psi^{-1}(\tau h)]}{h + \psi^{-1}(\tau \eta)} > \frac{\tau h}{(1+\tau)h} > \upsilon$$

we see that there must be points of X_{rsn} in the interval $(x_2 - \psi^{-1}(\tau h), x_2)$. Thus let x_3 be any point from X_{rsn} in that interval.

Again the set S_{x_3} satisfies a porosity requirement in the interval (x_3, x_0) : the interval $(x_0 - \theta h, x_0)$ has relative length at least

$$\frac{\theta h}{h + \psi^{-1}(\tau h)} > \frac{\theta}{1 + \tau} > p$$

so that there must be a point x_4 in S_{x_3} from the interval

$$(x_0 - \theta h, x_0).$$

We will obtain our contradiction by proving that

$$\frac{f(x_4) - f(x_1)}{x_4 - x_1} \leqq s$$

and since x_4 is evidently greater than x_2 this contradicts the definition of x_2 and our desired inequality will have been proved.

Putting our various computations together and using the Lipschitz condition, we obtain the inequalities

$$\begin{aligned} f(x_4) - f(x_1) &= [f(x_4) - f(x_3)] + [f(x_3) - f(x_2)] + [f(x_2) - f(x_1)] \\ &\leqq r(x_r - x_3) + \psi(|x_3 - x_2|) + s(x_2 - x_1). \end{aligned}$$

This leads to the inequalities

$$\begin{aligned} f(x_4) - f(x_1) &\leqq r(x_4 - x_3) + \tau h + s(x_2 - x_1) \\ &\leqq r(x_4 - x_2) + \tau h + s(x_2 - x_1) \\ &\leqq r(x_4 - x_2) + (s - r)(1 - \theta)h + s(x_2 - x_1) \\ &\leqq r(x_4 - x_2) + (s - r)(x_4 - x_2) + s(x_2 - x_1) \\ &\leqq s(x_4 - x_1) \end{aligned}$$

as required.

Thus we may conclude that for x_1 in X_{rsn}, $x_1 < x_0$, and sufficiently close to x_0 the inequality

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1} \leqq s$$

must hold. However we know that at x_0, $\overline{D}^- f(x_0) > s$; thus by the porosity lemma (2.6) the set X_{rsn} must have (ψ)-porosity positive at the point x_0. As this contradicts our earlier assertion we have exhibited each set X_{rsn} as having positive (ψ)-porosity at each point and the theorem is now proved.

As a corollary we may apply this theorem to certain density derivatives. We follow Zajiček [17] here in that we express the result as the existence of a type of path derivative along sets of upper density, but essentially this is nothing more than the observation that sets having positive lower (inner) density must have porosity less than 1.

(5.2) COROLLARY. Let f be a continuous function that satisfies an inequality of the form

$$|f(x) - f(y)| \leqq \psi(|x - y|) \qquad (|x - y| \leqq 1)$$

where ψ is as described in the theorem. Then at every point x with the possible exception of a σ-(ψ)-porous set there are measurable sets A_x and B_x each having upper density 1 at x and so that

$$\lim_{y \to x^+, y \in A_x} \frac{f(y) - f(x)}{y - x} = \bar{D}^+ f(x) = \bar{D}^- f(x) = \bar{D} f(x)$$

and

$$\lim_{y \to x^+, y \in B} \frac{f(y) - f(x)}{y - x} = \underline{D}^+ f(x) = \underline{D}^- f(x) = \underline{D} f(x) .$$

PROOF. If at a point x such a set A_x could not be found, then we can show that the set of points

$$\left[y : \frac{f(y) - f(x)}{y - x} < c < \bar{D} f(x) \right]$$

must have some positive lower density, and so must have porosity $p < 1$. This would then give that

$$\overline{PD}_p f(x) \; < \; c \; < \; \overline{D}f(x)$$

and we know from the theorem that the collection of such points has the asserted porosity requirement.

Let us conclude with several further applications of theorem (5.1) to the study of the differentiability properties of continuous functions. The fact that the porosity derivatives of a continuous function are residually the same as the ordinary Dini derivatives leads us to the following observations.

(5.3) THEOREM. Let f be continuous and have residually one of the four porosity derivates $\overline{PD}_p^{\,+}f(x)$, $\overline{PD}_p^{\,-}f(x)$, $\underline{PD}_p^{\,+}f(x)$, or $\underline{PD}_p^{\,-}f(x)$ finite for some value of $p < 1$ (p may depend on x). Then there is a dense open set G so that f is a.e. differentiable on G.

PROOF. By theorem (5.1) we know that residually the upper or lower bilateral derivates $\overline{D}f(x)$ or $\underline{D}f(x)$ must be finite. In fact then on a residual set one or other of the "sharp extreme derivates" $\overline{D}^{\$}f(x)$ or $\underline{D}^{\$}f(x)$ must be finite where these derivates are defined as

$$\overline{D}^{\$}f(x) \; = \; \inf_{\delta>0}\left[\sup \; \frac{f(y) - f(z)}{y - z} \; : \; y,z \in (x-\delta,x+\delta), y\neq z\right]$$

$$\underline{D}^{\$}f(x) \; = \; \sup_{\delta>0}\left[\inf \; \frac{f(y) - f(z)}{y - z} \; : \; y,z \in (x-\delta,x+\delta), y\neq z\right]$$

(This is a result of Bruckner and Goffman; see Bruckner [1,p.68]).

It is easy to check that if one of these derivates is finite at a point, then f must be a.e. differentiable in some neighborhood of that point. Consequently f is a.e. differentiable on some dense open set as required.

Similar arguments apply to prove the following theorem which is related to a theorem of Garg [7].

(5.4) THEOREM. Let f be a continuous, strictly increasing, singular function (i.e. $f'(x) = 0$ a.e.). Then there is a residual set at each point of which

$$\underline{PD}^{+}_{p} f(x) = \underline{PD}^{-}_{p} f(x) = 0$$

$$\overline{PD}^{+}_{p} f(x) = \overline{PD}^{-}_{p} f(x) = +\infty$$

for every p less than 1.

PROOF. For such a function f it is the case that at <u>every</u> point x the lower sharp derivate $\underline{D}^{\$}f(x)$ must vanish. For if not, then there would be an interval (c,d) in which the quotient

$$\frac{f(y) - f(z)}{y - z}$$

is bounded away from zero, and this cannot happen for this function f. In exactly the same way it follows that the uppper sharp derivate $\overline{D}^{\$}f(x)$ must be at every point $+\infty$. The theorem then follows directly from theorem (5.1) since residually the porosity derivates agree with the sharp derivates.

Another consequence of theorem (5.1) is that a continuous function which has at every point of an interval $[a,b]$ a path derivative (finite or infinite)

$$f_{E_x}{}'(x) = \lim_{y \to x, y \in E_x} \frac{f(y) - f(x)}{y - x}$$

along a path E_x that is nonporous at x (has right and left porosity 0 at x) must be in fact differentiable (finitely or infinitely) at each point of an open dense subset of the interval. If the path derivatives are given to be finite on the interval $[a,b]$ then there is a sequence of intervals $\{I_n\}$ whose union is dense in $[a,b]$ and such that f satisfies a Lipschitz condition,

$$|f(x) - f(y)| \leqq M_n |x - y| ,$$

on each interval I_n. For the special case of the sequential congruent derivatives this was first established by Laczkovich and Petruska [9].

By way of contrast note that it is generally less informative to have such a path derivative along sets that satisfy some density property. Indeed a typical continuous function will permit at almost every point x the existence of a path E_x that has upper density 1 at x and for which the derivative as above exists and is finite. But it follows from an observation of Jarník (see Bruckner [1,p.214]) that such functions may be nowhere differentiable or even nowhere approximately differentiable.

REFERENCES

[1] A.M.Bruckner, Differentiation of real functions, Lecture Notes in Math. No.659, Springer-Verlag (1978).

[2] A.M.Bruckner, R.J.O'Malley, and B.S.Thomson, Path derivatives: a unified view of certain generalized derivatives, Trans. Amer. Math. Soc., (1984), (to appear).

[3] A.Denjoy, Sur une propriété des series trigonométrique, Verlag v.d. G.V. der Wis-en Natuur. Afd., 30 October 1920.

[4] _______, Lecons sur le calcul des coefficients d' une serie trigonométrique, Part. II, Métrique et topologie d' ensembles parfaits et des fonctions, Gauthier-Villars, Paris (1941).

[5] E.P. Dolženko, Boundary properties of arbitrary functions, Math. USSR-IZV.1, (1967), 1-12.

[6] M.J. Evans and P.D. Humke, The equality of unilateral derivates, Proc. Amer. Math. Soc., 79(1980),609-613.

[7] K. Garg, On singular functions, Rev. Roumaine Math. Pures et Appl., 14(1969), 1441-1452.

[8] A. Khintchine, Recherches sur la structure des fonctions mesurables, Fund. Math. 9(1927),212-279.

[9] M. Laczkovich and G. Petruska, Remarks on a problem of A.M. Bruckner, Real Analysis Exchange 6 (1980/81),120-126.

[10] L. Mišík, Uber approximative derivierte Zahlen, Czech. Math J., 26 (101) (1976),579-583.

[11] H.H. Pu, J.D. Chen, and H.W. Pu, A theorem on approximate derivates, Bull. Inst. Math. Acad. Sinica, 2(1974), 87-91.

[12] S. Saks, Theory of the integral, Monografie Matematyczne 7, Warsawa-Lwow, (1937).

[13] P.D. Shukla, On the differentiability of monotone functions, Bull. Calcutta Math. Soc., 37(1945),9-14.

[14] B.S. Thomson, Some theorems for extreme derivates, J. London Math. Soc., (2), 27(1983), 43-50.

[15] _________, Derivation bases on the real line II, Real Analysis Exchange, 8 (1982/83), 278-442.

[16] W.H. Young, Oscillating succesions of continuous functions, Proc. London Math. Soc., (2)6(1908),298-320.

[17] L. Zajiček, On Dini derivatives of continuous and monotone functions, Real Analysis Exchange 7(1981-82),233-238.

Received March 5, 1984.