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On Foran's Conditions A(N), B(N) and (M)

Two continuous functions F_1 and F_2 are constructed, satisfying Lusin's condition (N) and Foran's condition B(2) on C (C = Cantor's ternary set), which are A(N) on no portion of C, and for no natural number N. Moreover, $F_1'(x) = -F_2'(x)$ a.e. on $[0,1]$, but F_1 and $-F_2$ do not differ by a constant. It is also shown that $G(x) = F_2(x)-(1/2)\varphi(x)$ (φ = Cantor's ternary function) fulfils Foran's condition (M), but does not fulfil Lusin's condition (N). Such a function was already obtained by Foran in [1], but in a more complicated way .

We recall the definitions of A(N) and B(N) given in [1], and that of (M) given in [2].

<u>Definition 1.</u> Given a natural number N and a set E, a function F is said to be B(N) on E if there is a number $M < \infty$ such that for any sequence $I_1, \ldots, I_k, \ldots$, of nonoverlapping intervals with $I_k \cap E \neq \emptyset$, there exist intervals J_{kn} , $n = 1,\ldots,N$, for which

$$B(F;E\cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^{N} I_k \times J_{kn} \quad \text{and} \quad \sum_k \sum_{n=1}^{N} |J_{kn}| < M .$$

(Here B(F;X) is the graph of F on the set X).

<u>Definition 2.</u> Given a natural number N and a set E, a function F is said to be A(N) on E if for every

$\varepsilon > 0$ there is a $\delta > 0$ such that if $I_1, \ldots, I_k, \ldots$ are nonoverlapping intervals with $E \cap I_k \neq \emptyset$ and $\sum |I_k| < \delta$, then there exist intervals J_{kn}, $n = 1, \ldots, N$, for which

$$B(F; E \cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^{N} I_k \times J_{kn} \quad \text{and} \quad \sum_k \sum_{n=1}^{N} |J_{kn}| < \varepsilon .$$

<u>Definition 3.</u> A continuous function F fulfils Foran's condition (M) if it is absolutely continuous on any set E on which it is of bounded variation.

Let $\mathcal{F}$ (respectively $\mathcal{B}$) be the class of all continuous functions F, defined on a closed interval I, for which there exist a sequence of sets E_n and natural numbers N_n, such that $I = \bigcup E_n$, and F is $A(N_n)$ (respectively $B(N_n)$) on E_n.

<u>Theorem 1. Let F be a real continuous function defined on a closed real set E. The function F belongs to $\mathcal{F}$ (respectively to $\mathcal{B}$) on E if and only if every closed subset of E contains a portion on which F is A(N) (respectively B(N)), for some natural number N.</u>

<u>Proof.</u> The proof is similar to that given in [3], pp.233-234.

Let $I = [0,1]$, and let C denote the Cantor ternary set, i.e., $C = \{x : x = \sum c_i/3^i$ with c_i taking the values 0 and 2 only$\}$. Each point $x \in C$ is uniquely represented by $\sum c_i(x)/3^i$. Let φ be Cantor's ternary function, i.e., $\varphi(x) = \sum c_k(x)/2^{k+1}$, for each $x\in C$. Then φ is continuous on C and, by extending φ linearly on each interval contiguous to C, one has φ defined and continuous on I. Let

$$F_1(x) = \sum c_{2k-1}(x)/4^k \quad \text{and} \quad F_2(x) = (1/2)\sum c_{2k}(x)/4^k,$$

for each $x\in C$. Then F_1 and F_2 are continuous on C and, by extending F_1 and F_2 linearly on each interval contiguous to C, one has F_1 and F_2 defined and continuous on I. Clearly

(1) $\varphi(x) = F_1(x) + F_2(x)$ on I

and

$$(2) \qquad F_1(x) = \begin{cases} (1/2)F_2(3x) & , x\in [0, 1/3] \\ x - (1/6) & , x\in (1/3 , 2/3) \\ (1/2)+(1/2)F_2(3x-2) & , x\in [2/3 ,1] . \end{cases}$$

<u>Theorem 2.</u> 1) F_1 <u>and</u> F_2 <u>fulfil Lusin's condition</u> (N) <u>on</u> I .

2) F_1 <u>and</u> F_2 <u>are</u> A(N) <u>on no portion of</u> C, <u>and for no natural number</u> N.

3) F_1 <u>and</u> F_2 <u>are</u> B(2) <u>on</u> C.

<u>Proof.</u> 1) $F_1(C)$ can be covered with 2^n intervals each of length at most $(1/3)(1/4^n)$, and $|F_1(C)| = 0$. $F_2(C)$ can be covered with 2^n intervals, each of length at most $(2/3)(1/4^n)$ and $|F_2(C)| = 0$.

2) We show that F_1 and F_2 do not belong to $\mathfrak{F}$ on I, hence by Theorem 1, F_1 and F_2 are A(N) on no portion of C, for no natural number N. Suppose on the contrary that F_2 belongs to $\mathfrak{F}$ on I. By (2) it follows

that also F_1 belongs to $\mathcal{F}$ on I, hence $F_1+F_2 \in \mathcal{F}$.
This contradicts (1).

3) Let $[a,b] \subset I$, $a,b \in C$. Then there is a largest interval (a_1,b_1) (and only one), excluded in the Cantor ternary process, such that $[a_1,b_1] \subset [a,b]$. Suppose that (a_1,b_1) is excluded at the nth step. Then

$$a_1 = \sum_{i=1}^{n} c_i/3^i + \sum_{i=1}^{\infty} 2/3^{n+i}\ , \text{ with } c_i=0 \text{ or } c_i=2 \text{ for}$$

$i < n$ and $c_n=0$. We have $a_1-a \leqslant 1/3^n$ and $b-b_1 \leqslant 1/3^n$.

If $x \in [a,a_1] \cap C$, then $c_i = c_i(x)$, for each $i = 1,\dots,n$. Hence

(3) $\qquad F_2(a_1)-F_2(x) \geqslant 0$ and $F_1(a_1)-F_1(x) \geqslant 0$.

Let $F_2(x_0) = \inf\{F_2(x) : x \in [a,a_1] \cap C\}$. Then $F_2([a,a_1] \cap C) \subset [F_2(x_0),F_2(a_1)] = J_1$ and by (3) one has

(4) $\qquad |J_1| \leqslant \varphi(a_1) - \varphi(a)$.

If $x \in [b_1,b] \cap C$, then $c_i = c_i(x)$, for each $i = 1,\dots,n-1$ and $c_n(x) = 2$. Hence

(5) $\qquad F_2(x)-F_2(b_1) \geqslant 0$ and $F_1(x)-F_1(b_1) \geqslant 0$.

Let $F_2(x_1) = \sup\{F_2(x) : x \in [b_1,b] \cap C\}$. Then $F_2([b_1,b] \cap C) \subset [F_2(b_1),F_2(x_1)] = J_2$ and by (5), one has

(6) $\qquad |J_2| \leqslant \varphi(b) - \varphi(b_1)$.

By (4) and (6), $F_2([a,b] \cap C) \subset J_1 \cup J_2$ and $|J_1| + |J_2| \leqslant \varphi(b) - \varphi(a)$.

Remark 1. 1) Theorem 2 shows that the Banach-Zarecki theorem ([3], pp.227) is not valid when AC and VB are replaced by A(N) and B(N) respectively. 2) The class $\mathfrak{F}$ is strictly contained in $\mathfrak{B} \cap (N)$. 3) F_1 and $-F_2$ satisfy (N), have equal derivatives a.e. and do not differ by a constant.

Remark 2. Let (J_i^p), $i = 1, \ldots, 2^{p-1}$ be the excluded middle thirds in Cantor's ternary process, from the pth step, numbered from left to right. We have $3J_i^{p+1} = J_i^p$, $i = 1, \ldots, 2^{p-1}$ and

$$J_{i+2^{p-1}}^{p+1} = (2/3) + J_i^p \quad , i = 1, \ldots, 2^{p-1} \quad .$$

Theorem 3. The functions F_1 and F_2 are not primitives in the Foran sense.

Proof. Suppose on the contrary that there is a continuous function G on I, belonging to $\mathfrak{F}$, such that $G'_{ap}(x) = F'_2(x)$ a.e. on I. Then there is a continuous function h on I, with $h(0) = 0$, which is constant on each interval contiguous to C, such that

(7) $\quad G(x) = F_2(x) + h(x) \quad .$

By (2) and (7), $F_1(x) + (1/2)h(3x)$ belongs to $\mathfrak{F}$ on $[0, 1/3]$. By (7), it follows that

(8) $\quad \varphi(x) + (1/2)h(3x) + h(x) = 0$ on $[0, 1/3]$.

For let $H(x) = G(x) + (1/2)G(3x) = \varphi(x) + (1/2)h(3x) + h(x)$, $x \in [0, 1/3]$. Then $H'_{ap} = 0$ a.e. and $G \in \mathfrak{F}$ implies $H \in \mathfrak{F}$ and $H \equiv C$. Since $H(0)=0$, (8) follows.

Since $\varphi(x) = (1/2)\varphi(3x)$, (8) becomes

(9) $(\varphi(x)/2 + h(x)) + (1/2)(\varphi(3x)/2 + h(3x)) = 0$,

for each $x \in [0, 1/3]$. Since

(10) $F_2(x) = F_2(x + 2/3)$, $x \in [0, 1/3]$,

we have $h(x+ 2/3) - h(x) = a$ for all $x \in [0, 1/3]$. For let $R(x) = G(x+ 2/3) - G(x) = h(x+ 2/3) - h(x)$, $x \in [0, 1/3]$. Since $G'_{ap}(x) = F'_2(x)$ a.e., $R'_{ap}(x) = 0$ a.e. and $R \in \mathcal{F}$ implies R constant. But $h(1/3) = h(2/3)$ and $h(0+ 2/3) - h(0) = a$ gives $h(2/3) = a$. Also $h(1/3 + 2/3) - h(1/3) = a$ so $h(1) = 2a$. Thus $h(x) = -(1/2)\varphi(x)$ on $[1/3 , 2/3]$. By Remark 2, (9) and (10), $h(x) = -(1/2)\varphi(x)$ on the closure of each interval contiguous to C and, by the continuity of h, we have that $h(x) = -(1/2)\varphi(x)$ on I. Thus $G(x) = F_2(x) - (1/2)\varphi(x)$ on I. Moreover, for each $x \in C$,

$G(x) = 1/6 - (1/2)(\sum (c_{2k-1}(x) + c_{2k}(x)/2 + 1)/4^k)$.

Hence $G(C) = [-1/3 , 1/6]$, and so G does not satisfy Lusin's condition (N). Contradiction.

<u>Remark 3.</u> The above function $G(x) = F_2(x) - (1/2)\varphi(x)$ on I is an example of a continuous function which does not satisfy Lusin's condition (N) but satisfies Foran's condition (M).

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