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On Typical Bounded Functions in the Zahorski Classes

Each of Zahorski's M_i classes [5] is closed under uniform convergence. Thus each class of bounded M_i functions on the interval [0,1] is a complete metric space under the sup norm. We can then investigate the behavior of a typical function, that is, belonging to a residual subset. We let I=[0,1].

Ceder and Pearson [3] show that the typical bounded Darboux Baire 1 function (that is, bounded M_1 function):

1 has an infinite derived number on both sides at every point

2 has both $+\infty$ and $-\infty$ as derived numbers at each point.

We will show that:

3 the typical bounded Darboux Baire 1 function has every real number as a derived number at every point

4 the three results above have direct analogues in each of the other Zahorski classes.

All functions will be real valued with domain contained in [0,1]. We use juxtaposition to indicate the intersection of two or more sets of functions. Thus, the bounded (b) Baire 1 functions are denoted by bB_1 and the bounded Darboux Baire 1 functions by bDB_1. $C^-(f,x)$ and $C^+(f,x)$ are the left and right cluster sets of f at x. For f in DB_1, the left and right cluster sets are intervals with f(x) in $C^-(f,x) \cap C^+(f,x)$ for all x. The associated sets of f are sets of the form $E_a(f)=\{x:f(x)>a\}$ and $E^a(f)=\{x:f(x)<a\}$ for a real. A function f is Baire 1 if each associated set is an F_σ set (or f is a pointwise limit of continuous functions. The Lebesgue measure of A is denoted by $\lambda(A)$. Zahorski's M_i classes (i=0,1,..,5) are defined as follows. A set E is in class M_i if E is an F_σ set and:

i=0 every x in E is a bilateral accumulation point of E

i=1 every x in E is a bilateral condensation point of E

i=2 for x in E and $\delta>0$, $\lambda((x-\delta,x)\cap E)>0$ and $\lambda((x,x+\delta)\cap E)>0$

i=3 for x in E and any sequence $\{I_n\}$ of intervals converging to x with $\lambda(I_n\cap E)=0$ for all n, $\lim_{n\to\infty}\lambda(I_n)/\text{dist}(x,I_n)=0$

i=4 if there exist sets K_n and positive numbers r_n such that $E=\cup K_n$ and for every x in K_n and $c>0$ there is an $\varepsilon(x,c)>0$ such that $\lambda(E\cap(x+h,x+h+k))/|k|>r_n$ for all h and k satisfying $hk>0$, $h/k<c$, and $|h+k|<\varepsilon(x,c)$

i=5 every x in E is a point of density of E $(d(E,x)=1)$.

A function f is in M_i (i=0,1,...,5) if each associated set is in class M_i. It has been shown that $M_0=M_1=DB_1$ and $M_5=A$ the approximately continuous functions. (Zahorski [5] or see Bruckner [2])

In the rest of this paper, the five classes will thus mean M_1 through M_5. We will make use of the fact that $M_1\supset M_2\supset\ldots\supset M_5$. For x in E, $\{x\}\subset_i E$ $(i\neq 4)$ will mean E satisfies the ith Zahorski condition at x. That is, when considering M_1, $\{x\}\subset_1 E$ means x is a bilateral condensation point of E. The context will make clear which classes are under consideration. For sets A and E, $A\subset_i E$ will mean $\{x\}\subset_i E$ for all x in A. Thus if f is in class M_5, $E_0(f)\subset_5 E_0(f)$ simply states that $d(E_0(f),x)=1$ for all x in $E_0(f)$.

<u>Lemma 1</u> For each i, bM_i is a complete metric space under the sup norm.

Proof: Consider i fixed. Suppose $f_n \to f$ where f_n is in bM_i for all n. Then f is in bDB_1 and it suffices to show that each associated set is in M_i. Let $E=E_a(f)$ and $E_k=E_{(a+1/k)}(f)$. For each k, pick n_k so that $\|f_{n_k}-f\|<1/2k$. Then $E_k \subset E_{(a+1/2k)}(f_{n_k}) \subset E$. Thus $E=\cup E_k=\cup E_{(a+1/2k)}(f_{n_k})$ is in M_i. $E^a(f)$ is similar.

The next lemma will be used in the proofs of later theorem.

<u>Lemma 2</u> For each i, the set A_i of all f in bM_i such that f is continuous on some subinterval is a first category F_σ set in bM_i.

Proof: Consider i fixed. Let $\{q_1,q_2,\ldots..\}$ be an enumeration of the rationals in I. For $q_n<q_m$ define $F_{n,m}$ to be the set of all f in bM_i such that f restricted to $[q_n,q_m]$ is continuous. Then A_i is the countable union of all such sets $F_{n,m}$. It remains to show that each $F_{n,m}$ is closed and nowhere dense in bM_i. Clearly each $F_{n,m}$ is closed. Fix n and m. Suppose $f \in F_{n,m}$ and $\varepsilon>0$. We can pick an h in $bM_5=bA$ so that

1) $0 \leq h \leq \varepsilon$

2) $h=0$ on $I-(q_n,q_m)$

3) h is not continuous on (q_n,q_m).

(See Zahorski [5], Bruckner [2], or Agronsky [1].)
Then it is easy to see that $g=f+h$ is in bM_i, $\|f-g\|=\varepsilon$, and g is not in $F_{n,m}$. Thus the complement of $F_{n,m}$ is dense and we are done.

We state two theorems of Ceder and Pearson mentioned in the introduction.

<u>Theorem A</u> The class of all functions in bDB_1 (bM_1) having an infinite derived number on each side at every point is a residual G_δ set.

Theorem B The class of all functions in bDB_1 (bM_1) having both $+\infty$ and $-\infty$ as derived numbers at each point is a residual G_δ set.

The proof of Theorem 1 below is a simplification of the proof of Theorem A, making it applicable to all five classes at once. We use much of the same notation as in [3].

Theorem 1 For each i, the class of all functions in bM_i having an infinite derived number on each side at every point is a residual G_δ set in bM_i.

Proof: Throughout the proof, we consider i fixed. Let A_R (respectively A_L) be all f in bM_i for which there is an x in $[0,1)$ (resp. $(0,1]$) such that f has no infinite derived number on the right (resp. left) at x. It suffices to show that A_R is a first category F_σ set. For n a natural number, θ and δ rational with $\delta>0$ and $0<\theta<\pi/2$, let $A(\theta,\delta,n)$ consist of all f in bM_i for which there is an x in $[0,1-1/n]$ such that $\tan\theta\geq|(f(z)-f(x))/(z-x)|$ for all $x<z<x+\delta$. Then A_R is the countable union of all such $A(\theta,\delta,n)$. It remains to show that each $A(\theta,\delta,n)$ is a closed nowhere dense set.

Fix θ, δ, and n. It is easy to see that $A(\theta,\delta,n)$ is closed. We say that f has "property A at x" if there is a z in $(x,x+\delta)$ such that $\tan\theta<|(f(z)-f(x))/(z-x)|$. If f has property A at every x in a set E, then we say f has "property A on E". Note, if f has property A on $[0,1)$, then f is in the complement of $A(\theta,\delta,n)$. Thus, it suffices to show that the functions with property A on $[0,1)$ form a dense set. For any point (x,y), let

$$K(x,y)=\{(u,v):x<u<x+\delta \text{ and } \tan\theta<|(v-y)/(u-x)|\}.$$

(The definitions and notation are from [3].)

Let $f \in A(\theta,\delta,n)$ and $\varepsilon>0$. By Lemma 2 we can pick h in bM_i so that h has a dense set of discontinuities and $\|f-h\|<\varepsilon$.

Let $Z=\{z:K(z,h(z))\cap h=\phi\}\cup\{(1,h(1))\}$. It is easy to see that Z is closed and that $\bar{h}|Z$ is continuous. By our choice of h, $\text{int}(Z)=\phi$. We then cover Z with finitely many intervals on which the oscillation of h is small and insert a steep saw-tooth function over each interval to obtain a g in bM_i such that $\|h-g\|<\varepsilon$ and g has property A on [0,1). See the proof of Lemma 4 in [3] for the insertion. Then $\|f-g\|<2\varepsilon$ and we are done.

The simplification of the proof is in our choice of the function h, making $\text{int}(Z)=\phi$.

Theorem A is then a special case of our Theorem 1. This is not the case for our next result. Our proof of the analogue of Theorem B for bM_i where $i\geq 2$ does not give Theorem B as a special case. Although we again borrow notation and follow the outline of the proof of Theorem B in [3], our proof will not work for bM_1.

<u>Theorem 2</u> For each $i\geq 2$, the class of all functions in bM_i having both $+\infty$ and $-\infty$ as derived numbers at each point is a residual G_δ set in bM_i.

Proof: We again consider i fixed. The proof is similar to that of Theorem 1. It suffices to show that, for $\delta>0$, and $0<\theta\leq\pi/2$, the set of all f in bM_i for which there is an x in I such that $\tan\theta\geq(f(z)-f(x))/(z-x)$ for all $0<|z-x|<\delta$ is a closed nowhere dense set. Let this set be $A'(\theta,\delta)$. We say f has property A' on I if $K'(x,f(x))\cap f\neq\phi$ for all x where

$$K'(x,y)=\{(u,v):0<|x-u|<\delta \text{ and } \tan\theta<(v-y)/(u-x)\}.$$

If f has property A' on I, then f is in the complement of $A'(\theta,\delta)$. It is easy to see that each $A'(\theta,\delta)$ is closed. We must show that the complement is dense. As before, fix θ and δ, let $f\epsilon A'(\theta,\delta)$, and let $\varepsilon>0$. Pick h in bM_1 so that $\|h-f\|<\varepsilon$ and h has a dense set of discontinuities. Let $Z=\{z:K'(z,h(z))\cap h=\phi\}$. It is easy to see that Z is closed and that $h|Z$ is continuous so $\text{int}(Z)=\phi$. Thus Z is nowhere dense. To avoid an easy added step, assume 1 is not isolated in Z. Let $I-Z=\cup(a_j,b_j)$ where each (a_j,b_j) is a component of I-Z. Since $h|Z$ is continuous, we may cover Z with finitely many intervals $[x_k,y_k]$, $1\le k\le n$, so that:

(1) $x_1<y_1<x_2<y_2<\ldots<x_n<y_n$

(2) $y_k-x_k<\delta_1=\min(\delta/2,\varepsilon/8\tan\theta)$ for each k

(3) if x and y are in $[x_k,y_k]\cap Z$, then $|h(x)-h(y)|<\varepsilon/4$

(4) $x_k\epsilon Z$ for each k, $y_k\epsilon Z$ if $(x_k,y_k)\cap Z\neq\phi$, and $y_k\notin Z$ otherwise.

Observe that if x is a right limit (left limit) of Z, then h is right (left) continuous at x. In fact, if x and y are in $[x_k,y_k]\cap Z$ for some k, then $|h(r)-h(s)|\le\varepsilon/4+2\delta_1\tan\theta<\varepsilon/4+\varepsilon/4=\varepsilon/2$ for all r and s between x and y. This results from the fact that the graph of $h|[x,y]$ is contained in the parallelogram $P(x,y)$ formed by the vertical lines through (x,0) and (y,0) and the lines with slope $\tan\theta$ through (x,h(x)) and (y,h(y)).

For each k, at least one of the following occurs:

C1 $[x_k,y_k]\cap Z=\{x_k\}$

C2 there is an isolated point, z_k, of Z in (x_k,y_k)

C3 there is a component, (a_{j_k},b_{j_k}), of I-Z in (x_k,y_k).

Note that C2 and C3 are not mutually exclusive. If C2 occurs, then there is a $\gamma_k>0$ such that $(z_k-\gamma_k,z_k+\gamma_k)\cap Z=\{z_k\}$.

For $1 \le k \le n$, we define A_k in one of three possible ways. Let A_k be $E_{h(x_k)-\varepsilon/2}(h) \cap (x_k, y_k)$ if C1 occurs, $(z_k, z_k+\gamma_k)$ if C2 occurs, and (a_{j_k}, m_k) where $m_k = (a_{j_k} + b_{j_k})/2$ if C3 occurs but C2 does not. Let $A_0 = \cup A_k$. Then $A_0 \in M_i$. Since h is approximately continuous almost everywhere (Thm 5.2[2]), we can pick $A \subset A_0$ so that $\lambda(A) = \lambda(A_0)$, h is approximately continuous at every x in A, and $A \subset_5 A$.

Let T be a countable dense subset of h. By repeated application the Luzin-Menchoff Theorem (Thm 6.4[2]), we can construct an F_σ set $W_A = \cup W_m \subset A\text{-dom}(T)$ so that $W_A \subset_5 W_A$ and $W_1 \cap (x_k, y_k) \ne \phi$ for each k. Then, by Zahorski [5] (or see Bruckner [2] or a special case of a construction of Agronsky [1]), we can construct an h_1 in bM_5 such that $h_1 = 0$ on $I - W_A$, $h_1 \ge 0$, h_1 is usc (h_1 is continuous at every x in $I - W_A$), $\|h_1\| = \varepsilon$, and $h_1 | W_1 = \varepsilon$.

For $1 \le k \le n$, let B_k be the empty set if C1 occurs, $(z_k - \gamma_k, z_k)$ if C2 occurs, and (m_k, b_{j_k}) if C3 occurs and C2 does not. Let $B_0 = \cup B_k$. As above, we can pick a set $B \subset B_0$ with the same properties as $A \subset A_0$, and an F_σ set $W_B = \cup W'_m \subset B\text{-dom}(T)$ so that $W_B \subset_5 W_B$ and $W'_1 \cap (x_k, y_k) \ne \phi$ if $B_k \ne \phi$. We then construct h_2 in bM_5 such that $h_2 = 0$ on $I - W_B$, $h_2 \le 0$, h_2 is lsc (h_2 is continuous at every x in $I - W_B$), $\|h_2\| = \varepsilon$, and $h_2 | W'_1 = -\varepsilon$. Observe that $W_A \cap W_B = \phi$.

Let $g = h + h_1 + h_2$. Then $\|f - g\| < 2\varepsilon$, and $g \in bB_1$. Our next lemma will show that $g \in bM_i$. Let C_f (resp. A_f) be the set of continuity (resp. approximate continuity) points of a function f.

<u>Lemma 3</u> Let $F \in bM_i$. If $G \in bA$, G is usc (resp. lsc), $G \ge 0$ (resp. $G \le 0$), $C_G = \{x | G(x) = 0\}$, and $I - A_F \subset C_G$, then $F + G \in bM_i$.

Proof: Fix i. We prove the result for G usc. Let $E = E_a(F+G)$. Then $E = E_a(F) \cup (E \cap E_0(G))$. It is easy to see that $E \cap E_0(G)$ is an M_5

set since $E_0(G)\subset A_F$. Thus E is the union of two M_i sets and $E\epsilon M_i$.

Now let $E=E^a(F+G)$. For $x\epsilon E$, we can pick a rational r so that $F(x)<r<a$ and $r-F(x)<a-(F+G)(x)$. We use the upper semi-continuity of G to pick an interval J with rational endpoints so that $x\epsilon J$ and $E^r(F)\cap J\subset E$. Since E is the countable union of such sets, $E\epsilon M_i$. The case of G lsc is similar. This completes the proof of the lemma.

We now show that g has property A' on I by four cases.

1 If $x\epsilon I-\cup[x_k,y_k]$, then $g(x)=h(x)$. Since $K'(x,h(x))\cap h\neq\emptyset$ and T is dense in h, $K'(x,g(x))\cap g\neq\emptyset$ ($g=h$ on dom(T)).

2 Suppose $x\epsilon[x_k,y_k]$ for some k and C1 occurs. If $x=x_k$, then by our construction $g(x)=h(x)$ and $K'(x,g(x))\cap g\neq\emptyset$. If $x\epsilon(x_k,y_k]\cap W_A$, then $g(x)>h(x)$. Since T is dense in h, h(x) is a left cluster value of T. Thus $K'(x,g(x))\cap g\neq\emptyset$. If $x\epsilon(x_k,y_k]-W_A$, then $g(x)=h(x)$ and the argument of 1 above applies.

3 Suppose $x\epsilon[x_k,y_k]$ and C2 occurs. If $x\epsilon Z\cap[x_k,z_k]$, then g is above $P(x_k,y_k)$ on a subset of $(z_k,z_k+\gamma_k)$, so $K'(x,g(x))\cap g\neq\emptyset$. If $x\epsilon Z\cap[z_k,y_k]$, then g is below $P(x_k,y_k)$ on a subset of $(z_k-\gamma_k,z_k)$, so $K'(x,g(x))\cap g\neq\emptyset$. If $x\epsilon W_A\cap(x_k,y_k)$, then, as in 2 above, $K'(x,g(x))\cap g\neq\emptyset$. A similar argument applies to $x\epsilon W_B\cap(x_k,y_k)$. If $x\epsilon[x_k,y_k]-(W_A\cup W_B\cup Z)$, then the argument in 1 above applies.

4 If $x\epsilon[x_k,y_k]$ and C3 occurs but C2 does not, then an argument similar to that in 3 above applies.

Thus g has property A' on I and this finishes the proof of the theorem.

Note that the case of bM_1 does not follow from the proof of Theorem 2 since it is possible that $f\epsilon bM_1$, $E_a(f)\neq\emptyset$, but $\lambda(E_a(f))=0$.

We prove Theorem 3 for the case bM_1 and then indicate how simple modifications in the proof yield the analogues for bM_i where $i \geq 2$.

<u>Theorem 3</u> The class of all functions in bDB_1 having every real number as a derived number at every point is a residual set.

Proof: Let X be the class of all functions in bDB_1 having both $+\infty$ and $-\infty$ as derived numbers at every point, a residual G_δ set by Theorem B. Let N be the class of all functions in X having every real number as a derived number at every point. The proof will show that X-N is first category in X.

We need some terminology to use later in the proof. For any point (x,y) let o(x,y) be the open half-ray $\{(x,v):v<y\}$ and let O(x,y) be the open half-ray $\{(x,v):v>y\}$. For θ, β, and δ positive rationals with θ and β angle measures less than π such that $\theta+\beta>\pi$, let $r(\theta,\beta,\delta,x,y)$ be the open set of all (u,v) such that:

1) (u,v) is in o(x,y) or
2) $x<u<x+\delta$ and the angle between the line segment joining (x,y) to (u,v) and o(x,y) is less than θ or
3) $x-\delta<u<x$ and the angle between the line segment joining (x,y) to (u,v) and o(x,y) is less than β.

$R(\theta,\beta,\delta,x,y)$ is similarly defined using O(x,y) instead of o(x,y).

Let $y(\theta,\beta,\delta)$ be the set of all f in X with $r(\theta,\beta,\delta,x,f(x))\cap f=\phi$ for some x in I, and $Y(\theta,\beta,\delta)$ the set of all f in X with $R(\theta,\beta,\delta,x,f(x))\cap f=\phi$ for some x in I. Then X-N is the countable union of all such $y(\theta,\beta,\delta)$ and $Y(\theta,\beta,\delta)$.

We define $b_f(x)=\inf(C^-(f,x)\cup C^+(f,x))$. Let $z(\theta,\beta,\delta)$ be the set of all f in X with $r(\theta,\beta,\delta,x,b_f(x))\cap f=\phi$ for some x in I. We call such an x a z-value of f. $Z(\theta,\beta,\delta)$ is similarly defined using

$t_f(x)=\sup(C^-(f,x)\cup C^+(f,x))$ and $R(\theta,\beta,\delta,x,t_f(x))$. In the rest of the proof θ, β, and δ are fixed. It is easy to see that $y(\theta,\beta,\delta)$ is a subset of $z(\theta,\beta,\delta)$. It remains to show that $z(\theta,\beta,\delta)$ is a closed nowhere dense subset of X. A similar argument will apply to $Y(\theta,\beta,\delta)\subset Z(\theta,\beta,\delta)$.

Lemma 4 $z(\theta,\beta,\delta)$ is closed in X.

Proof: Suppose $f_n\to f$ uniformly and $f_n\in z(\theta,\beta,\delta)$ for all n. Let x_n be a z-value for f_n. Since we can pick a convergent subsequence, let us assume that $x_n\to x$ in I. It is easy to see that $b_{f_n}(x_n)\to b_f(x)$. The fact that each x_n is a z-value of f_n forces x to be a z-value of f. Thus $f\in z(\theta,\beta,\delta)$ and the set is closed.

Lemma 5 The complement of $z(\theta,\beta,\delta)$ is dense in X.

Before proving the lemma we make an observation. If f is in $z(\theta,\beta,\delta)$ and x and y are z-values of f, then $|x-y|\geq\delta$. Thus the set of z-values of f is a finite set.

Proof of Lemma 5: Suppose $f\in z(\theta,\beta,\delta)$, $\varepsilon>0$, and x is a z-value of f. We will construct a g in X such that $\|f-g\|\leq\varepsilon$, x is not a z-value of g, and the set of z-values of g is contained in the set of z-values of f. Essentially, the construction of g eliminates one z-value of f.

Since $b_f(x)$ is in $C^-(f,x)\cup C^+(f,x)$, we may assume that $b_f(x)$ is in $C^-(f,x)$. Let T be a countable dense subset of f. We can then pick a c-dense in itself F_σ subset of f, say E, so that:

1 $E\cap T=\phi$

2 $x\in cl(dom(E))-dom(E)$

3 $cl(dom(E))\subset(x-\delta,x]$

4 $f(x_n)\to b_f(x)$ if $x_n\in dom(E)$ and $x_n\to x$.

By Theorem 1 of [3] there is an h in bDB_1 so that h=f on I-dom(E) and $f>h\geq f-\varepsilon$ on dom(E). By using the construction in the proof of that theorem we can easily pick h so that $r(\theta,\beta,\delta,x,b_f(x))\cap h\neq\phi$ and if $y\notin$dom(E) and $x_n\to y$ with $x_n\in$dom(E), then $f(x_n)-h(x_n)\to 0$. Observe that $b_h(x)=b_f(x)$. Let ℓ be the line through $(x,b_h(x))$ such that the angle between ℓ and $o(x,b_h(x))$ is $(\beta+\pi-\theta)/2$. We define

$$g(z)=\begin{cases} f(z)\,(=h(z)) & \text{on } I-(x-\delta,x) \\ \max(h(z),\ell(z)) & \text{on } (x-\delta,x). \end{cases}$$

Observe that g=f on I-dom(E). It is easy to see that g is in bDB_1, $b_g(x)=b_f(x)$, and $r(\theta,\beta,\delta,x,b_g(x))\cap g\neq\phi$.

Claim 1 $g\in X$.

Proof: If $z\in$dom(E), then $g(z)<f(z)$. Since E misses T and T is dense in f, $+\infty$ is a derived number on the right at z and $-\infty$ on the left. If $z\notin$dom(E), then g(z)=f(z) and g has the same derived numbers as f since g=f on dom(T) and T is dense in f. Thus $g\in X$.

Claim 2 The z-values of g are also z-values of f and x is not a z-value of g.

Proof: By our construction, x is not a z-value of g. Suppose y is not a z-value of f. We consider two cases.

1 If $y\in$I-cl(dom(E)), then $b_g(y)=b_f(y)$ and $r(\theta,\beta,\delta,y,b_f(y))\cap f\neq\phi$. Since $g\leq f$, $r(\theta,\beta,\delta,y,b_g(y))\cap g\neq\phi$. Thus y is not a z-value of g.

2 If $y\in$cl(dom(E)), then by our construction $y\in(x-\delta,x)$ and $b_g(y)\geq\ell(y)$ so that $(x,b_g(x))$ is in $r(\theta,\beta,\delta,y,b_g(y))$. Thus $r(\theta,\beta,\delta,y,b_g(y))\cap g\neq\phi$ and y is not a z-value of g.

This verifies the claim.

By making this modification near each z-value of f, we can construct $g\in X-z(\theta,\beta,\delta)$ so that $\|f-g\|\leq\varepsilon$. This completes the proof of Lemma 5 and the theorem.

Theorem 4 For each $i \geq 2$, the class of all functions in bM_i having every real number as a derived number at every point is a residual set in bM_i.

Proof: The proof is very similar to that of Theorem 3. The analogue of the set E in the proof of Lemma 5 should be chosen as we picked W_A in Theorem 2, with $E \subset E^{f(x)+\varepsilon/2}(f) \cap (x-\delta, x)$. We then proceed to construct the function h as we did h_1 or h_2 in Theorem 2. The rest of the proof remains unaltered.

An open question is whether or not the set of typical functions of Theorem 3 or 4 is a G_δ set. Other properties found to be typical of bDB_1 functions could be candidates for the bM_i case. See Ceder and Pearson [4] for a survey of such properties and interesting candidates not decided at this time.

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