 Real Analysis Exchange Vol. 9 (1983-84)

Ewa Lazarow, Institute of Mathematics, 90-238 Łodz,
Stefan Banach 22, Poland.

Selective differentiation of typical continuous functions

The notion of selective derivative was introduced by R.J. O'Malley [5] who presented some interesting theorems and problems on selective derivatives.

R.J. O'Malley has proved that for every continuous function there exists a selection S and a point x_o such that f has a selective derivative at x_o. His question was: If f is a continuous function, for how large a set A does there have to exist a selection S with respect to which f has a selective derivative at every point of A.

In this paper we shall show that the set A has Lebesgue measure zero and is of the first category.

To simplify the later computation we shall use $[a,b]$ to denote the closed interval having endpoints a and b regardless of whether $a > b$ or $a < b$. By a selection we mean an interval function $s([a,b])$ for which $a < s([a,b]) < b$ holds for every $0 \le a < b \le 1$. We define the lower selective derivative $_sf'(x)$ of the function $f(x)$ by

$$_sf'(x) = \liminf_{h\to 0} \frac{f(s([x,x+h])) - f(x)}{s([x,x+h]) - x} .$$

It should be clear from the above definition how we would define the upper selective derivative, $^sf'(x)$, selective derivative $sf'(x)$ and one-sided selective derivatives.

Let $C[0,1]$ denote the class of continuous functions on $[0,1]$ furnished with the "sup" norm $\|f\| = \max_{0 \leq t \leq 1} |f(t)|$.

By "typical" continuous functions we mean those which form a residual subset in the complete metric space $C[0,1]$. Now we shall give several basic theorems and refer the reader to [5] and [4] as important sources.

<u>Theorem A</u>: (See [5], Lemma 1.)

Let $f:[0,1] \to R$ and S be a fixed selection.

Let $P_n = \{x : \dfrac{f(s([x,x+h])) - f(x)}{s([x,x+h]) - x} > 0$ for all h with $|h| < \frac{1}{n}\}$.

If $x < y$ and both belong to P_n and if $y - x < \frac{1}{n}$, then $f(x) < f(y)$. Hence f is of bounded variation on P_n.

<u>Theorem B</u>: (See [5], Lemma 2.)

Let $f:[0,1] \to R$ and let S be a fixed selection. Let P_n be defined as above and let $\mathcal{C}\ell\ P_n$ be its closure. Let $x < y$ be any two points such that

(i) the distance between x and y is less than $\frac{1}{n}$

(ii) there is a decreasing sequence $\{x_k\}$ of points of P_n converging to x

(iii) there is an increasing sequence $\{y_i\}$ of points of P_n converging to y

(iv) $\mathrm{Min}\ [{}_sf'(x), {}_sf'(y)] > -\infty$.

Then $f(x) < f(y)$.

Theorem C: (See [4].)

If $f:[0,1] \to R$ has a selective derivative $sf'(x)$ for a given selection S, then the set of points of continuity of $sf'(x)$ is everywhere dense in $[0,1]$.

Theorem D: (See [2].)

Let f be defined on a perfect set P. Suppose f satisfies condition (i) or condition (ii) below.

(i) f has the property of Baire on P.

(ii) f is measurable with respect to some non-atomic measure for which Lusin's theorem holds and such that $\mu(P) > 0$.

Then there exists a non-empty perfect set $Q \subset P$ such that $f|_Q$ is differentiable at all $x \in Q$ (infinite derivatives allowed).

Theorem E: (See [1], page 60.)

Let f be a continuous function defined on R and let $-\infty < \alpha < \infty$. If the set $\{x : D^+f(x) \geq \alpha\}$ is dense in R and if there exists an $x_0 \in R$ such that $D^+f(x_0) < \alpha$, then the set $\{x : D^+f(x) = \alpha\}$ has cardinality c, the cardinality of R.

Theorem 1.

Let $f:[0,1] \to R$ be a Lebesgue measurable function. Then there is a selection S and a set P of cardinality c such that f has a selective derivative (possibly infinite) at all $x \in P$.

Proof.

By Theorem D we have that there is a perfect set $Q \subset [0,1]$ such that $f|_Q$ is differentiable at all $x \in Q$. Let $[a,b] \subset [0,1]$.

If $(a,b) \cap Q \neq \emptyset$, then let $s([a,b])$ be any point $x_0 \in Q \cap (a,b)$. If $(a,b) \cap Q = \emptyset$, then let $s([a,b]) = \frac{a+b}{2}$. We denote the set of bilateral limit points of Q by P. Then $Q = P \cup C$ where C is at most countable. Let $x_0 \in P$ and $h_n \to 0^+$. Then $(x,x+h_n) \cap Q \neq \emptyset$ for every n and $x_{h_n} = s([x,x+h_n]) \in (x,x+h_n) \cap Q \subset Q$ and $x_{h_n} \to x_0^+$.

Hence

$$\lim_{n\to\infty} \frac{f(s([x_0,x_0+h_n])) - f(x_0)}{s([x_0,x_0+h_n]) - x_0} = \lim_{n\to\infty} \frac{f(x_{h_n}) - f(x_0)}{x_{h_n} - x_0} =$$

$$= f|_Q'(x_0).$$

Analogously, if $h_n \to 0^-$, then

$$\lim_{n\to\infty} \frac{f(s([x_0-h_n,x_0])) - f(x_0)}{s([x_0-h_n,x_0]) - x_0} = f|_Q'(x_0).$$

Since the sequences $\{h_n\}$ were arbitrary,

$$sf^-(x_0) = sf'(x_0) = f|_Q'(x_0).$$

This completes the proof of Theorem 1.

The next example shows that the set P of cardinality c cannot be arbitrary even if we assume that f is an absolutely continuous function.

Example 1.

Let A be a subset of $[0,1]$ such that for each interval $[a,b] \subset [0,1]$ $\mu([a,b] \cap A) > 0$ and $\mu([a,b] \cap ([0,1] \setminus A)) > 0$.

Let $f(x) = \int_0^x \chi_A(t)\,dt$.

Then $f'(x) = \chi_A(x)$ almost everywhere.

Let $B = \{x \in [0,1] \ f'(x) = \chi_A(x)\}$ and $P = [0,1] \setminus B$. Then $\mu(P) = 0$. Let $\alpha \in (0,1)$. Then $\{x : D^+f(x) \geq \alpha\} \supset \{x : f'(x) = 1\}$ is dense on $[0,1]$ and there exists an $x_0 \in [0,1]$ such that $D^+f(x_0) = f'(x_0) = 0 < \alpha$. Hence $\{x : D^+f(x) = \alpha\}$ has cardinality c. Then $P \supset \{x : D^+f(x) = \alpha\}$ has cardinality c. We suppose that there exists a selection S such that f has a selective derivative at all points $x \in P$. Then the function f has a selective derivative at all points $x \in [0,1]$ and $sf'(x) = \chi_A(x)$ almost everywhere. Let $[a,b] \subset [0,1]$. Then $\operatorname{osc}_{[a,b]} sf'(x) = 1$ and consequently sf' is an everywhere discontinuous function which contradicts Theorem C.

Theorem 2.

There exists a continuous function $f : [0,1] \to R$ such that for every selection S the set of points at which the selective derivative (possibly infinite) of the function f exists with respect to S is of measure zero and of first category. In fact the set of such functions is a residual subset of $C[0,1]$.

Proof.

Let f be a continuous, nowhere approximately differentiable function on $[0,1]$ (See [3].) Let S be a fixed selection. Let P be the set of those points in which a selective derivative exists (possibly infinite) with respect to S. Let $P_1 = \{x \in [0,1] : {}_S f'(x) > -\infty\}$ and $P_2 = \{x \in [0,1] : {}^S f'(x) < \infty\}$. Then $P \subset P_1 \cup P_2$. Let $P_1^n = \{x \in [0,1] : \frac{f(s([x,x+h])) - f(x)}{s([x,x+h]) - x} > -n$ for all h with $|h| < \frac{1}{n}\}$. Then $P_1 \subset \bigcup_n P_1^n \subset \bigcup_n \mathcal{Cl}\ P_1^n$ where $\mathcal{Cl}\ P_1^n$ denotes closure of P_1^n. Suppose that there is n_o such that $\mu(\mathcal{Cl}\ P_1^{n_o}) > 0$. Let $g(x) = f(x) + n_o x$. Then the set $\mathcal{Cl}\ P_1^{n_o}$ will be precisely the set $\mathcal{Cl}\ P_n$ of Theorem A for the function $g(x)$. Let Q denote the set of bilateral limit points of $\mathcal{Cl}\ P_1^{n_o}$ and $Q_i = Q \cap [\frac{i-1}{2n}, \frac{i}{2n}]$ for $i = 1,2,\ldots 2n$. Then by Theorem A and Theorem B g is increasing on Q_i for each i and therefore g is measurable and of generalized bounded variation on $\mathcal{Cl}\ P_1^{n_o}$. Hence the function g is approximately derivable at almost all points of $\mathcal{Cl}\ P_1^{n_o}$ which contradicts the assumption of the function $f(x) = g(x) - n_o x$. Therefore for every n $\mu(\mathcal{Cl}\ P_1^n) = 0$ and $\mathcal{Cl}\ P_1^n$ is nowhere dense set. Hence the set P_1 is of measure zero and of first category.

Let $h(x) = -f(x)$. Then ${}_S h(x) = -{}^S f(x)$ and $P_2 = \{x \in [0,1]\ \ {}^S f'(x) < \infty\} = \{x \in [0,1]\ \ {}_S h'\ x) > -\infty\}$. Hence $\mu(P_2) = 0$ and P_2 is a set of first category and also $P \subset P_1 \cup P_2$.

Because the set of nowhere approximately differentiable functions is a residual subset of $C[0,1]$ ([3]), the theorem is proved.

Theorem 3.

Let $f : [0,1] \to R$ and $K \subset [0,1]$ satisfy the following conditions:

(i) if $x \in K$ and $f'(x)$ exists, then $|f'(x)| < \infty$

(ii) for every $x \in K$ $D_L f(x) \cap D_R f(x) \neq \emptyset$ where $D_L f(x)$ $(D_R f(x))$ denotes the set of left-sided (right-sided) derived numbers at x.

(iii) There exists a number n_0 such that $K \cap K^{(n_0)} = \emptyset$ where $K^{(n_0)}$ denotes the set of limit points of $K^{(n_0-1)}$ and $K^0 = K$.

Let $g : K \to R$ be a function such that for every $x \in K$ $g(x) \in D_L f(x) \cap D_R f(x)$ and $|g(x)| < \infty$. Then there is a selection S such that at all $x \in K$ $sf'(x) = g(x)$.

The assumptions (i), (ii), (iii) are necessary -

(i) For every set K such that $\mu(K) = 0$ there exists the continuous function f such that $f'(x) = \infty$ at all $x \in K$. (See [1], page 229.)

(ii) By [5], $sD_L f(x) \subset D_L f(x)$ and $sD_R f(x) \subset D_R f(x)$ where $sD_L f(x)$ $(sD_R f(x))$ denotes the set of left-sided (right-sided) selective derived numbers of x. If $D_L f(x) \cap D_R f(x) = \emptyset$, then $sD_L f(x) \cap sD_R f(x) = \emptyset$.

(iii) Let P denote the set of bilateral limit points of the Cantor set C. The set C is of measure zero and of the first category. We define

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \setminus P \\ 0 & \text{if } x \in P \end{cases} .$$

For every natural n $C \cap C^{(n)} = C$. If $x \in [0,1] \setminus C$, then $f'(x) = 1$. If $x \in C \setminus P$, then $D_L f(x) \cap D_R f(x) = \{1\}$ and if $x \in P$, then $D_L f(x) \cap D_R f(x) = \{0\}$. If there exists the selection S such that the function f is selectively differentiable with respect to S at all $x \in C$, then $sf'(x) = 1$ at $x \in C \setminus P$ and $sf'(x) = 0$ at $x \in P$. Of course $sf'(x) = 1$ at $x \in [0,1] \setminus C$ and the selective derivative exists and it is finite at all points of $[0,1]$. This is impossible because f is not a Darboux function nor is it Baire 1. (See [5], Theorem 11.)

<u>Proof of Theorem 3</u>.

Let $x_o \in K$. Then $g(x_o) \in D_L f(x_o) \cap D_R f(x_o)$ and there are two sequences $\{x_n\}$, $\{y_n\}$ such that $x_n \to x_o$ and for every n $x_n > x_o$ and $y_n \to x_o$, $y_n > y_o$ and

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_o)}{x_n - x_o} = \lim_{n\to\infty} \frac{f(y_n) - f(x_o)}{y_n - x_o} = g(x_o).$$

Since for every $k \geq 2$ $K^{(k)} \subset K^{(k-1)}$, $K \supset K \cap K' \supset K'' \cap K \supset \ldots$ $\ldots \supset K \cap K^{(n_o-1)} \supset K \cap K^{(n_o)} = \emptyset$. Let $[a,b] \subset [0,1]$.

$$s([a,b]) \in \{a_n > a : a_n \to a \quad \lim_{n\to\infty} \frac{f(a_n) - f(a)}{a_n - a} = g(a)\} \cap (a,b)$$

if and only if

1) $a \in K$ and $b \notin K$

or

2) $a \in K \cap K'$ and $b \notin K \cap K'$

or

3) there is a natural number p such that $a \in K \cap K^{(p)}$

$a \notin K \cap K^{(p+1)}$ and there is a natural number $s < p$ such that $b \in K \cap K^{(s)}$ and $b \notin K \cap K^{(s+1)}$.

$$s([a,b]) \in \{b_n < b : b_n \to b \quad \lim_{n\to\infty} \frac{f(b_n) - f(b)}{b_n - b} = g(b)\} \cap (a,b)$$

if and only if

4) $b \in K$ and $a \notin K$

or

5) $b \in K \cap K$ and $a \notin K \cap K'$

or

6) there is a natural number p such that $b \in K \cap K^{(p)}$, $b \notin K \cap K^{(p+1)}$ and there is a natural number $s < p$ such that $a \in K \cap K^{(s)}$ and $a \notin K \cap K^{(s+1)}$.

If (1) - (6) aren't satisfied, then let $s([a,b]) = \frac{a+b}{2}$. Let $x_o \in K$ and $K = K^o$. Then there is a number $p \in \{0,1,2,\dots n_o-1\}$ such that $x_o \in K \cap K^{(p)}$ and $x_o \notin K \cap K^{(p+1)}$. Therefore, there is a $\delta > 0$ such that $(x_o - \delta, x_o + \delta) \cap K \cap K^{(p)} = \{x_o\}$. Let $|h| < \delta$. If $0 < h < \delta$, then $[x_o, x_o + h] \subset [x_o, x_o + \delta)$ and $x_o + h \in K \cap K^{(s)}$ where $s < p$ or $x_o + h \notin K$ if $p = 0$, and $x_o + h \notin K \cap K^{(p)}$. Therefore, $s([x_o, x_o + h]) \in$ $\{x_n^o > x_o : x_n^o \to x_o \quad \lim_{n\to\infty} \frac{f(x_n^o) - f(x_o)}{x_n^o - x_o} = g(x_o)\}$. If $-\delta < h < 0$, then $[x_o - h, x_o] \subset (x_o - \delta, x_o]$, $x_o - h \in K \cap K^{(\ell)}$ where $\ell < p$ or $x_o - h \notin K$ if $\ell = 0$, and $x_o - h \notin K \cap K^{(p)}$. Therefore

$$s([x_o - h, x_o]) \in \{x_n^o < x_o : x_n^o \to x_o, \quad \lim_{n\to\infty} \frac{f(x_n^o) - f(x_o)}{x_n - x_o} = g(x_o)\}.$$

Hence

$$sf'(x) = \lim_{h\to 0} \frac{f(s([x_o, x_o+h])) - f(x_o)}{s([x_o, x_o+h]) - x_o} = g(x_o)$$

and the theorem is proved.

REFERENCES

[1] A.M. Bruckner - Differentiation of real function. Lecture Notes in Mathematics, 659, Springer-Verlag, New York 1978.

[2] A.M. Bruckner, J.Ceder and M.Weiss - On the differentiability structure of real function. Trans. Amer. Math. Soc. 142 (1969), 1-13.

[3] V. Jarnik - Sur la dérivabilité des fonctions continuous. Spisy Privadov. Fak. Univ. Korlovy 129 (1934), 3-9.

[4] M. Laczkovich - On the Baire class of selective derivatives. Acta.Math. Sci. Hung. 29 (1977), 99-105.

[5] R.J. O'Malley - Selective derivates. Acta. Math. Sci. Hung. 29 (1977), 77-97.

Received April 15, 1984.