ON FUNCTIONAL COMPLEXITY AND SUPERPOSITIONS OF FUNCTIONS

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The theory of functional complexity deals with the relative structure characteristics of functions, and one of its aims is to develop criteria which will make it possible to assign numerical measures to the complexity of given classes of functions. The general notion of functional complexity can be traced to nomography, but the present day interest in the subject and the direction its research has taken grew out of Hilbert's Problem 13 of his famous Paris lecture of 1900 [11]. Here we are interested in functional complexity as it applies to continuous functions of two or more variables. Their relative complexity will be decided according to their representability in terms of superpositions, using as a measure of comparison that two functions have the same complexity if they can be composed in a finite number of steps from functions belonging to the same class or classes. We see at once that the answers we can expect will depend on the choices of superpositions and on the conditions imposed on the classes of functions allowed in these superpositions: for example, these may be required to be of a certain form, or to satisfy algebraic or differentiability conditions. The purpose of this talk is to introduce this general circle of ideas through a discussion of selected results in this area. A comprehensive introduction to the subject from a variety of perspectives can be found in [1, 5, 16, 23, and 30].

We begin with an example: The elementary functions

$$\sin xy \qquad zx^y \qquad \frac{xy + xz + yz}{(x^2 + y^2 + z^2)^{1/2}}$$

can be represented in terms of a finite superposition of continuous functions of

one variable and the binary operation of addition. This follows from trivial identities and the observation that

$$xy = \frac{1}{4}(x + y)^2 - \frac{1}{4}(x - y)^2$$
$$= u(a(x) + b(y)) + v(c(x) + d(y)),$$

where

$$u(t) = -v(t) = \frac{1}{4} t^2$$
$$a(t) = b(t) = c(t) = -d(t) = t .$$

Hence, the above functions can be said to have the same functional complexity if our measure is representability with continuous functions of one variable. If instead we impose the condition that the superposition functions must be algebraic, then these particular functions will no longer share the same complexity.

In its most general form, the problem under consideration can be conveniently described by means of a commuting diagram: Consider metric spaces X, Y, and T , and a given mapping $f : X \to Y$. We are interested in finding mappings $H : X \to T$ and $G : T \to Y$, such that f can be replaced with the superposition $G \circ H$:

(1) $$f = G \circ H .$$

Diagram 1
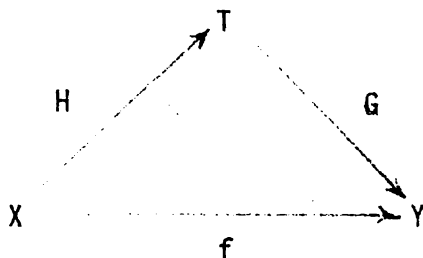
Representations such as (1) are of particular interest when the mapping H is a fixed embedding of X into T , and when (1) is valid for a sufficiently large class of functions f. We will retain this particular

aspect of generality in our discussion, but the setting will be restricted

to the Banach spaces $C(E^n)$ of continuous functions defined on the n-fold

cartesian product $E \times E \times \dots \times E = E^n$ of unit intervals $E = [1, 2]$ and

having range on the real line R. The norm will be the uniform norm, and

n-dimensional Euclidean space will be designated by R^n. With this restriction,

we replace the above commuting diagram with the following:

Diagram 2

$$R^N$$

H   G $(N \geq n \geq 2)$

$$E^n \longrightarrow R$$

f
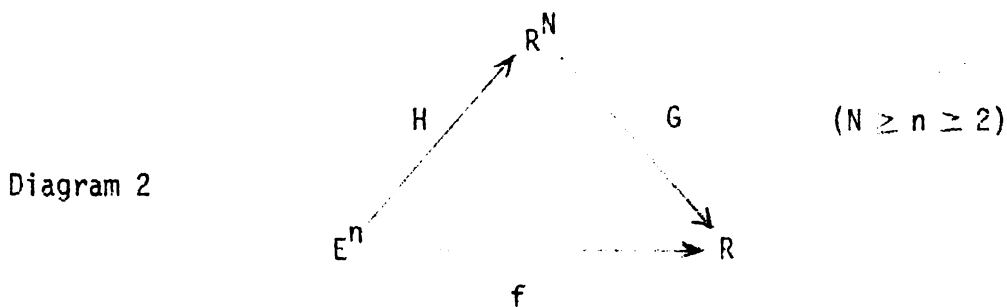
Beginning with a class $F_n \subset C(E^n)$ of mappings

$$f : E^n \longrightarrow R$$

we wish to determine an integer N and a continuous embedding

$$H : E^n \longrightarrow R^N$$

for which the following is true: For each function $f \in F_n$ there is a

continuous mapping

$$G : R^N \longrightarrow R$$

such that (1) holds. The classes F_n are assumed to be sufficiently large,

such as the entire space $C(E^n)$, all its polynomials or analytic functions,

or its p-times continuously differentiable functions. This and the requirement

that H be independent of the class F_n impose an important restriction

on the choices available for the mappings H, since for any such class F_n

the following is true:

Given arbitrary points $\underline{x} \neq \underline{y}$ in E^n , then there exists a function $f \in F_n$ such that $f(\underline{x}) \neq f(\underline{y})$.

This means that the class F_n separates the points of E^n , in the sense that any two distinct points have distinct images for some function $f \in F_n$. Since $G \circ H(\underline{x}) = G \circ H(\underline{y})$ whenever $H(\underline{x}) = H(\underline{y})$, it is necessary for the fixed mapping H to also separate the points of E^n . This implies that the embedding H is a homeomorphism of E^n into R^N . This important aspect of the problem will be touched on only peripherally here, and for a fuller discussion we refer to $[22,23,26,27]$.

What we are leading up to is the remarkable theorem of Kolmogorov which states that for each natural number $n \geq 2$ the functions of $C(E^n)$ are finite superpositions of functions of $C(E)$, but to better understand the implications of this result and how it relates to the question of functional complexity, we shall look briefly at the genesis of the general problem.

Nomography, you may recall, deals with the problem of finding graphical solutions to functional equations through suitable parametrizations. Interest in this area of mathematics peaked in the late 1800's, when d'Ocagne published a definitive treatise on the subject [17]. For nomographic constructions, however, it is generally necessary that no more than two parameters be used at any given stage of the process. This tells us, for example, that the roots of polynomial equations of degrees 1,2,3, and 4 -- when regarded as functions of the coefficients -- can be obtained through nomographic con- structions, because the method of radicals reduces the solutions into steps which require at most two parameters at a time. In fact, consider the general polynomial equation

$$f^n + x_1 f^{n-1} + x_2 f^{n-2} + \ldots + x_{n-1} f + x_n = 0 \quad (n \geq 2)$$

where we have already eliminated the leading coefficient using division.

The term containing f^{n-1} can be eliminated with the transformation

$$f = g - x_1/n \; ;$$

the constant term x_n can be replaced by 1 again using division. Less known today is the fact that also the terms involving f^{n-2} and f^{n-3} can be eliminated with the help of the so-called Tschirnhaus transformations which generalize the method of radicals. Thus, the general polynomial equation of degree 6 can be reduced to the form

$$f^6 + xf^2 + yf + 1 = 0$$

by using only algebraic operations, and the roots are therefore obtainable through nomographic constructions (as are the roots of the general polynomial equation of degree 5). I mention in passing that the above identity for the product xy can be used repeatedly to express the roots of polynomial equations of degrees $n \leq 4$ in terms of finite superpositions of algebraic functions of one variable and addition. Now, using algebraic reductions, the general polynomial equation of degree 7 can be reduced to the equivalent form

(2) $$f^7 + xf^3 + yf^2 + zf + 1 = 0.$$

But in this form, the roots are still functions of three variables, so that nomographic construction is not possible.

Hilbert has realized that for the purposes of nomography it was not necessary to restrict the reductions to alebraic operations, and that any continuous transformations would do. He thus attacked the seventh degree polynomial equation with more general tools. Convincing himself that it was not possible to eliminate one of the three remaining coefficients in equation (2), he conjectured in the above mentioned Problem 13 that "the equation of the seventh degree $f^7 + xf^3 + yf^2 + zf + 1 = 0$ is not solvable with the help of any continuous functions of only two arguments." It is

interesting to note that Hilbert's formulation of the problem is in two parts, one purely algebraic and one purely analytic, and that he included the problem in the algebra section of his lecture -- a choice which affirms the origin of the problem and also indicates his own outlook. He returned to the 13th problem in 1927 with the surprising result that the general polynomial equation of 9th degree can be reduced to the equivalent form

$$f^9 + xf^4 + yf^3 + zf^2 + tf + 1 = 0 \ .$$

He also discussed more explicitly the notion of functional complexity in this paper, but it is important to note that his method for obtaining the above reduction was algebraic[12].

For those not familiar with this literature, it may be of interest to mention that in the formulation of Problem 13 Hilbert made the incorrect statement "I have satisfied myself by a rigorous process that there exist analytical functions of three arguments x, y, z which cannot be obtained by a finite chain of functions of only two arguments." It is clear, however, that this was only a slip on Hilbert's part, and that he meant analytic functions; it was well known to him that every function of three variables is a finite superposition of functions of two variables.

Bieberbach attributed this slip to the fact that 13 was an unlucky number, and this was somewhat prophetic on his part: In 1930 he published what he thought to be a solution to Hilbert's conjecture by showing that there are polynomials of three variables which are not the uniform limit of superpositions of polynomials of two variables. His argument rested on the following observation: A general polynomial of degree n in three variables has n^3 arbitrary coefficients, whereas k superpositions of polynomials of degree n in two variables have kn^2 arbitrary coefficients. Hence, if we fix k, then n can be taken to be so large that $n^3 > kn^2$, implying that the coefficients must satisfy certain algebraic conditions.

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It can be shown, however, that there exist coefficients which do not satisfy any such relationships. Bieberbach made a curious mistake in this paper [3]: he applied the Weierstrass approximation theorem to a compact domain while letting the boundaries tend to infinity. This error was discovered by Kamke and acknowledged by Bieberbach in a note published in 1934 [4]. In this note Bieberbach proposed a new lemma which would have established the correctness of his result, but the proof of the lemma eluded him. This was just as well, because eventually it also turned out to be incorrect.

Now, we know that smoothness conditions provide a useful measure of functional complexity, and it was Hilbert's notion that the number of variables of a function is also a useful measure in the following sense: Consider a continuous function f of n variables, and a class S_m of continuous functions of $m \leq n$ variables. If the function f can be represented as a finite superposition of functions of S_m, and if m is the least natural number for which this is true, then we can use m as a measure of the functional complexity of f relative to the class S_m. We can expect more interesting answers when the classes S_m have suitable restrictions, and imposing smoothness conditions is clearly in tenor with Hilbert's Problem 13. The earliest significant result along these lines was obtained by Ostrowski in 1920 [20]. He proved that the analytic function

$$\zeta(x,y) = \sum_{0=1}^{\infty} x^n/n^y$$

cannot be obtained as a finite superposition of infinitely differentiable functions of one variable and algebraic functions of any number of variables. The most profound result in this direction was proved by Vituškin in 1954 [30]: Let $q = p + a$, where p is a non-negative integer, and $0 < a \leq 1$; let W_q^n be the class of p times continuously differentiable functions of

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n variables defined on a closed and bounded region of R^n , and whose p^{th} partial derivatives belong to the class Lip(a). If we use $\frac{n}{q}$ $(q \geq 1)$ as the measure of functional complexity of the class W_q^n , then not all

functions with complexity $\frac{n}{q}$ can be represented with superpositions of functions with complexity $\frac{n'}{q'} < \frac{n}{q}$.

The above theorem of Vituškin shows that the number of variables in combination with a differentiability condition can serve as a useful measure of functional complexity. It also establishes an inevitable descent in the smoothness of the functions used in superpositions, in the sense that a decrease in the number of variables forces a decrease in smoothness. Thus, for example, not all functions of three variables with $p \geq 1$ continuous partial derivatives can be represented with superpositions of functions of two variables having p continuous partial derivatives. A number of other important results in this general direction have been obtained by Vituškin and Henkin (see, for example, [31] and [32]); some still open questions can be found in [24].

We now turn our attention to Kolmogorov's theorem [14]. In its simplest form it can be stated as follows:

Theorem 1.

For each natural number $n \geq 2$ there exist monotonic increasing functions in each variable

(3) $$H_q(\underline{x}) = \sum_{q=1}^{n} h_{pq}(x_p) \qquad q = 1, 2, \ldots, 2n + 1 ,$$

$h_{pq} \in C(E)$, with the property that for each function $f \in C(E^n)$ there are continuous functions $g_q \in C(E)$ such that

(4) $$f(\underline{x}) = \sum_{q=1}^{2n+1} g_q \circ H_q(\underline{x}) .$$

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We observe at once that this theorem refutes Hilbert's conjecture, since it demonstrates that all classes of functions $C(E^n)$ for $n \geq 2$ have the same functional complexity as the class $C(E)$. It also tells us something very important about uniform continuity. Namely, that the increase in degrees of freedom with increasing n does not increase the functional complexity, implying that the 'worst' functions of $C(E)$ are as 'bad' as the 'worst' functions of $C(E^n)$. Whether a similar statement can be made about continuous functions which are not necessarily uniformly continuous remains an open question. Doss [8] has obtained a result in this direction: Using Ostrand's theorem [19] which states that Kolmogorov's theorem is true for compact metric spaces of finite covering dimension, he was able to show that continuous functions defined on an open cube or open ball or all of R^n can be represented with 4n superpositions of the form (4), except that the fixed functions H_q are functions of n variables not expressible in terms of continuous functions of fewer variables. This theorem thus lacks the most important property of the Kolmogorov type superpositions, and it would be interesting to determine if Kolmogorov's theorem can be established without compactness.

An important improvement in Kolmogorov's construction was obtained by Fridman [9] who showed that the fixed functions h_{pq} in (3) can be constructed to belong to the class Lip(1). It was observed by Lorentz [15] that the $2n + 1$ continuous functions g_q in Theorem 1 can be replaced by a single continuous function g; the author has shown that instead of (3) we can use functions

$$(5) \qquad H_q^*(\underline{x}) = \sum_{q=1}^{n} \lambda_p h_q(x_p) \qquad q = 1, 2, \ldots, 2n + 1,$$

where the functions h_q $C(E)$ are monotonic increasing and belong to the class Lip(1) and $\lambda_1, \ldots, \lambda_n$ are rationally independent constants [22,25].

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I mention in passing that the $2n + 1$ functions h_q can be replaced by linear translations of a single continuous function with the same properties [25]. As far as the number of summands $2n + 1$ in formula (4) is concerned, it turned out to be the best possible: Using Kolmogorov's construction, Doss [7] was able to show that this is the case when $n = 2$. In a more general setting, Sternfeld has obtained very interesting results which provide great insight into Kolmogorov's theorem, and among the results he obtained was a theorem stating that the smallest number of summands in formula (4) is $2n + 1$ for all $n \geq 2$ [28].

Let us now set

$$H = (H_1, \ldots, H_{2n+1}).$$

With this notation, we see that formula (4) is of the general form $f = G \circ H$, so that Theorem 1 can be interpreted by means of the commuting diagram 2 with $N = 2n + 1$. Consider the subspace $L(E^{2n+1}) \subset C(E^{2n+1})$ of linear functions

$$(6) \qquad F(\underline{X}) = \sum_{q=1}^{2n+1} f_q(X_q)$$

and the system of parametric equations

$$(7) \qquad \begin{aligned} X_1 &= X_1(x_1, \ldots, x_n) \\ X_2 &= X_2(x_1, \ldots, x_n) \\ &\vdots \\ X_{2n+1} &= X_{2n+1}(x_1, \ldots, x_n) \end{aligned}$$

or, more specifically,

$$(8) \qquad X_q = \sum_{p=1}^{n} h_{pq}(x_p) \qquad q = 1, 2, \ldots, 2n + 1.$$

Either one of these systems of equations determines an embedding of E^n

into R^{2n+1}, and we also note that a substitution of (7) or (8) into

equation (6) gives rise to functions of $C(E^n)$. Thus, the systems of

equations (7) or (8) establish a mapping of a subset of $C(E^n)$ into

$L(E^{2n+1})$. For the sake of definiteness, we shall restrict our discussion

to (8).

In view of the above observations, we consider the functional

$$(9) \qquad \qquad \mu(f) = \inf_{f_q \in C(E)} \| f(\underline{x}) - F(\underline{X}) \|$$

where f is a given function of $C(E^n)$, $F \in L(E^{2n+1})$ is specified by

6), and $\underline{X} = (X_1, \ldots, X_{2n+1})$ is determined by (8). We note that this

is the setting of a standard problem in approximation theory, and in general

we expect the functional $\mu(f)$ to vanish only on a small subset of $C(E^n)$,

assuming, of course, that best approximations exist. This can be guaranteed,

however, for the functions which we are interested in. To summarize this

line of reasoning, we observe that we seek approximations to the functions

of $C(E^n)$ with the restrictions of linear functions of $L(E^{2n+1})$ to certain

homeomorphic images of E^n in R^{2n+1}. Kolmogorov's theorem tells us that

$\mu(f)$ will vanish identically on $C(E^n)$ if we make the right choice of

embeddings (8) of E^n into R^{2n+1}, and it is thus of interest to study

the properties of these embeddings and to characterize those which give the

desired result. We have already noted that continuity by itself is too

weak a condition when we study functional complexity, and it turns out that

perhaps not much in terms of structure theorems of these embeddings can be

expected. We mentioned their basic smoothness properties, and Kolmogorov's

construction scheme and its various variants give us insight into the nature

of these fixed functions. That not much more may be hoped for follows from

theorems of Hedberg [10] and Kahane [13] which state, roughly speaking, that

almost any choice will do.   In these papers the authors use category arguments, and they do not attempt to construct the functions involved. The functional (9) can be approached with a modification of the procedures developed by Diliberto and Straus in [6] (see also [5]), and it is possible that this could lead to an alternative constructive proof of Theorem 1 and its various refinements.

In concluding this talk, I would like to say a few words about convergence problems associated with representations with superpositions. We note that, as the many theorems of approximation theory demonstrate, it is very difficult to meaningfully relate functional complexity and uniform approximability since the uniform closure of even very restricted classes of functions is often too large to be useful. Also, even if we have a sequence

$$\mu_r(f) = \| f(\underline{x}) - F_r(\underline{X}) \|$$

such that

$$\lim_{r \to \infty} \mu_r(f) = 0$$

we are not guaranteed that the limit of the sequence $\{F_r\}$ will be of the desired form. This can be illustrated with a simple example:

Consider the product xy defined on E^2. This product can be obtained as the uniform limit

$$xy = \lim_{r \to \infty} \exp\left[\ln(x + \frac{1}{r}) + \ln(y + \frac{1}{r}) \right]$$

$$= \lim_{r \to \infty} g\left[a_r(x) + b_r(y) \right] \ .$$

The functions g, a, and b are continuous and strictly monotonic increasing. The function f(x,y) = xy is strictly monotonic in each variable, except on the coordinate axes. Yet, the limit of the sequence $\{ g[a_r(x) + b_r(y)] \}$ is not of the form $g[a(x) + b(y)]$ (see [21] and [29]).

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