REMARKS ON AGRONSKY'S THEOREM ON STRONG CONTAINMENT AND CONTINUITY ROADS

by

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In his paper Associated Sets and Continuity Roads [1], S. Agronsky gives an abstract analysis of a characterization of those functions in a class which remain in the class when composed with continuous functions. The classes he considers are defined by associated sets and the characterization is by means of continuity roads, where the associated sets and roads are determined by his concept of strong containment relations. In his paper, Agronsky gives sufficient conditions on a strong containment relation to yield his characterization theorem. At the 1983 Waterloo Symposium on Real Analysis, the question of the necessity of these conditions was raised. The present note investigates this question. It is shown that only a few of the conditions are necessary, though all of the conditions are natural in some sense when applied to the study of derivatives as Agronsky has shown in [1].

For ease of exposition we set forth some notation. The Lebesgue measure of a set A will be denoted by $\mu(A)$. The set of functions of Baire Class one will be denoted by $\mathbb{B}_1$. For each real-valued function f and real number a , we have the two associated sets

$$E_a(f) = \{ x \mid f(x) > a \} ,$$

and

$$E^a(f) = \{ x \mid f(x) < a \} .$$

For a set X, the power set is $\mathcal{P}(X) = \{ A \mid A \subset X \}$ and the set of all functions $f : X \to \mathbb{R}$ is $\mathbb{R}^X$.

Let α be a relation on $\mathcal{P}(\mathbb{R})$, that is $\alpha \subset \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})$. For $x \in \mathbb{R}$, $\mathcal{T}_\alpha[x]$ is the set of functions $f : \mathbb{R} \to \mathbb{R}$ which have the property that there is a set E such that $x \in E$, $E \alpha E$ and $f|E$ is continuous at x. The set of functions $f \in \mathbb{R}^\mathbb{R}$ for which $E_a(f) \alpha E_a(f)$ and $E^a(f) \alpha E^a(f)$ for all $a \in \mathbb{R}$ will be denoted by $\mathcal{T}_\alpha$. Of course, $\mathcal{T}_\alpha[x]$ and $\mathcal{T}_\alpha$ can be suitably modified to use functions whose domains are sets X contained in $\mathbb{R}$. We remark that Agronsky's concept of a strong containment relation is a relation on $\mathcal{P}(\mathbb{R})$ satisfying certain conditions.

For any $\mathcal{T} \subset \mathbb{R}^X$, we define a new class of functions comp $\mathcal{T}$ to be the set

$$\text{comp } \mathcal{T} = \{ f \in \mathbb{R}^X \mid g \circ f \in \mathcal{T} \text{ for each continuous } g : \mathbb{R} \to \mathbb{R} \} .$$

Clearly, comp $\mathcal{T} \subset \mathcal{T}$.

For later reference, we next state the theorem of S. Agronsky [1].

<u>Theorem</u>. Let α be a relation on $\mathcal{P}(\mathbb{R})$ which satisfies the following seven conditions.

A1: $E \alpha F$ and $F \subset G \Rightarrow E \alpha G$.

A2: $E \subset F$ and $F \alpha G \Rightarrow E \alpha G$.

A3: $E_n \propto F_n$ $(n = 1,2,\ldots)$ $\Rightarrow$ $\bigcup_{n=1}^{\infty} E_n \propto \bigcup_{n=1}^{\infty} F_n$.

A4: For each open set G , $G \propto G$.

A5: $E \propto E$ and G open $\Rightarrow$ $E \cap G \propto E \cap G$.

A6: $E \propto F$ $\Leftrightarrow$ for each $x \in E$, $\{x\} \propto F$.

A7: $E_{n+1} \subset E_n$ and $E_n \propto E_n$ $(n = 1,2,\ldots)$ and $\bigcap_{n=1}^{\infty} E_n = \{x\}$ $\Rightarrow$ there is a sequence δ_n decreasing to 0 such that

$$\{x\} \propto \left(\{x\} \cup \bigcup_{n=1}^{\infty} \{z \in E_n : |x-z| > \delta_n\}\right) .$$

Then, for $f \in \mathcal{B}_1$, the following statements are equivalent.

(1) $f \in T_\alpha[x]$ for all $x \in \mathbb{R}$.

(2) $f^{-1}[G] \propto f^{-1}[G]$ for each open set G .

(3) $g \circ f \in T_\alpha$ for each continuous function $g : \mathbb{R} \to \mathbb{R}$; that is, $f \in \text{comp } T_\alpha$.

In [1], Agronsky uses a different condition A7. It is easily shown that A1-A6 together with Agronsky's version of A7 are equivalent to the conditions A1-A7 given above. In passing, we remark that Agronsky's strong containment relations are those which satisfy the conditions A1, A2 and A3 above.

In Section 1, we show that the equivalence of (2) and (3) of Agronsky's Theorem is true in general. Section 2 is devoted to the implication (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (1) is discussed in Section 3. Finally, Section 4 is an investigation of the necessity of the conditions A1-A7 of Agronsky's Theorem.

1. The Equivalence of (2) and (3).

We will show that none of the conditions A1-A7 are needed to prove this equivalence. First, we give the definitions of two operations.

For each $\mathfrak{M} \subset \mathcal{P}(X)$ and each $\mathcal{T} \subset \mathbb{R}^X$ we define the operations C and A as follows:

$$C(\mathfrak{M}) = \left\{ f \in \mathbb{R}^X \mid E_a(f) \in \mathfrak{M} \text{ and } E^a(f) \in \mathfrak{M} \text{ for every } a \in \mathbb{R} \right\}.$$

$$A(\mathcal{T}) = \left\{ A \in \mathcal{P}(X) \mid \exists a \in \mathbb{R}, \exists f \in \mathcal{T} \ni A = E_a(f) \text{ or } A = E^a(f) \right\}.$$

We list some easily proved claims.

1.1. Claim. $A(C(\mathfrak{M})) \subset \mathfrak{M}$. If $0, X \in \mathfrak{M}$ then $0, X \in A(C(\mathfrak{M}))$.

1.2. Claim. $\mathcal{T} \subset C(A(\mathcal{T}))$.

1.3. Claim. $\mathfrak{M}_1 \subset \mathfrak{M}_2 \Rightarrow C(\mathfrak{M}_1) \subset C(\mathfrak{M}_2)$.

1.4. Claim. $\mathcal{T}_1 \subset \mathcal{T}_2 \Rightarrow A(\mathcal{T}_1) \subset A(\mathcal{T}_2)$.

1.5. Claim. $C(A(C(\mathfrak{M}))) = C(\mathfrak{M})$. I.e., $C \circ A \circ C = C$.

1.6. Claim. $A(C(A(\mathcal{T}))) = A(\mathcal{T})$. I.e., $A \circ C \circ A = A$.

We now prove our general proposition.

1.7. Proposition. Let $\mathfrak{M} \subset \mathcal{P}(X)$. The following are equivalent statements.

(i) $f \in \text{comp } C(\mathfrak{M})$.

(ii) $f^{-1}[G] \in \mathfrak{M}$ for each open set G.

(iii) $f^{-1}[G] \in A \circ C(\mathfrak{M})$ for each open set G.

Proof. That (ii) implies (iii) is an immediate consequence of Claim 1.1 above. We prove (iii) implies (i). Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and $a \in \mathbb{R}$. Then $E_a(g \circ f) = f^{-1}[E_a(g)] \in \mathfrak{M}$ and $E^a(g \circ f) = f^{-1}[E^a(g)] \in \mathfrak{M}$, since $E_a(g)$ and $E^a(g)$ are open sets. We finally prove

(i) implies (ii). Let G be an open set. There is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $E_0(g) = G$. Since $E_0(g \circ f) = f^{-1}[G]$ and $\text{comp } C(\mathfrak{M}) = \text{comp } C \circ A \circ C(\mathfrak{M})$ by Claim 1.5, we have $f^{-1}[G] \in A \circ C(\mathfrak{M})$.

We remark that though $\mathfrak{M}$ could be large, it is $A \circ C(\mathfrak{M})$ which is important for the determination of $f \in \text{comp } C(\mathfrak{M})$.

By assuming more about $\mathfrak{M}$, we are able to reduce the test to open intervals (a,b) of $\mathbb{R}$. Let $\mathfrak{M} \subset \mathcal{P}(X)$. We say $\mathfrak{M}$ satisfies the <u>condition</u> $\underline{U_0}$ if

$$U_0 : E_n \in \mathfrak{M} \ (n = 1,2,\ldots) \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathfrak{M} .$$

1.8. <u>Proposition</u>. Let $\mathfrak{M} \subset \mathcal{P}(X)$. If $\mathfrak{M}$ satisfies the condition U_0 then the following are equivalent.

(i) $f \in \text{comp } C(\mathfrak{M})$.

(ii) $f^{-1}[(a,b)] \in \mathfrak{M}$ for each open interval (a,b).

<u>Proof</u>. Each open set G of $\mathbb{R}$ is a countable union of open intervals. Hence Proposition 1.7 yields Proposition 1.8.

1.9. <u>Corollary</u>. Let α be a relation on $\mathcal{P}(\mathbb{R})$. Then $f \in \text{comp } T_\alpha$ if and only if $f^{-1}[G] \ \alpha \ f^{-1}[G]$ for each open set G. Moreover, if α satisfies the condition A3, then $f \in \text{comp } T_\alpha$ if and only if

$f^{-1}[(a,b)] \ \alpha \ f^{-1}[(a,b)]$ for each open interval (a,b)

<u>Proof</u>. Let $\mathfrak{M} = \left\{ A \in \mathcal{P}(\mathbb{R}) \mid A \ \alpha \ A \right\}$. Then $C(\mathfrak{M}) = T_\alpha$ and $\mathfrak{M}$ satisfies condition U_0 when α satisfies condition A3.

Clearly, the above Corollary is the equivalence of (2) and (3) in Agronsky's Theorem.

2. The Implication (1) ⇒ (2) of Agronsky's Theorem.

In the proof of the above stated implication, we will show that all of the conditions A1-A7 are not used. Agronsky has already observed that A7 is not used.

Let α be a relation on $\mathcal{P}(\mathbb{R})$. We say α satisfies the condition U if

$$U : \text{For any index set } \Lambda \text{ , } E_\lambda \ \alpha \ F \text{ for all}$$

$$\lambda \in \Lambda \text{ implies } \bigcup_{\lambda \in \Lambda} E_\lambda \ \alpha \ F \text{ .}$$

2.1. Claim. Let α be a relation on $\mathcal{P}(\mathbb{R})$. If α satisfies the condition A6 then α satisfies the condition U .

2.2. Theorem. Let α be a relation on $\mathcal{P}(\mathbb{R})$ satisfying the conditions A1, A5 and U . Then $f \in \mathcal{T}_\alpha[x]$ for all $x \in \mathbb{R}$ implies $f^{-1}[G] \ \alpha \ f^{-1}[G]$ for each open set G .

Proof. Suppose G is an open set and $f \in \mathcal{T}_\alpha[x]$ for each $x \in \mathbb{R}$. Let $x \in f^{-1}[G]$. There are sets E_x and H_x such that $x \in E_x \cap H_x$, $E_x \ \alpha \ E_x$, H_x is open and $E_x \cap H_x \subset f^{-1}[G]$. Conditions A1 and A5 yield $E_x \cap H_x \ \alpha \ f^{-1}[G]$. Finally, condition U gives

$$f^{-1}[G] = \bigcup_{x \in f^{-1}[G]} E_x \cap H_x \ \alpha \ f^{-1}[G] \text{ .}$$

The following theorem of Agronsky now follows from Claim 2.1.

2.3. Theorem. Let α be a relation on $\mathcal{P}(\mathbb{R})$ satisfying the conditions A1, A5 and A6. Then, $f \in T_\alpha[x]$ for all $x \in \mathbb{R}$ implies $f^{-1}[G] \,\alpha\, f^{-1}[G]$ for each open set G.

2.4. Example. Let α be defined by $E \,\alpha\, F$ if and only if E is an open set and $E \subset F$. Clearly α satisfies the condition U but not the condition A6.

We now continue to investigate some relationships between the eight conditions A1-A7 and U.

2.5. Claim. A1 + U ⇒ A3. Consequently, A1 + A6 ⇒ A3.

Proof. Suppose $E_n \,\alpha\, F_n$ $(n = 1,2,\ldots)$. Then condition A1 gives $E_n \,\alpha \bigcup_{n=1}^{\infty} F_n$ for each n. Condition U completes the proof.

2.6. Claim. A6 ⇒ A2.

3. The Implication (2) ⇒ (1) of Agronsky's Theorem.

We first prove the following theorem.

3.1. Theorem. Let α be a relation on $\mathcal{P}(\mathbb{R})$ satisfying the conditions A3, A5 and A7. Suppose $x \in \mathbb{R}$ and f satisfy $f^{-1}[G] \,\alpha\, f^{-1}[G]$ for each open set G containing $f(x)$. Then $f \in T_\alpha[x]$.

Proof. Let $G_n = \{ y \mid |y-f(x)| < 1/n \}$. Then $f^{-1}[G_n] \,\alpha\, f^{-1}[G_n]$ for $n = 1,2,\ldots$. By using condition A5, we can find $E_n \subset f^{-1}[G_n]$ such that

$\{x\} = \bigcap_{n=1}^{\infty} E_n$ and $E_{n+1} \subset E_n$, $E_n \alpha E_n$ $(n = 1,2,\ldots)$. From condition A7, there is a sequence δ_n decreasing to 0 such that the set

$$E = \left(\{x\} \cup \bigcup_{n=1}^{\infty} \left\{ z \in E_n \mid |x-z| > \delta_n \right\}\right)$$

has $\{x\} \alpha E$. From condition A5, we have

$$\left\{ z \in E_n \mid |x-z| > \delta_n \right\} \alpha \left\{ z \in E_n \mid |x-z| > \delta_n \right\}$$

for each $n = 1,2,\ldots$. Hence, condition A3 implies $E \alpha E$. Let $\epsilon > 0$ and choose $n_0 > \epsilon^{-1}$. If $z \in E$ and $0 < |x-z| < \delta_{n_0}$ then, for $n \leqslant n_0$, $z \notin \left\{ z \in E_n \mid |x-z| > \delta_n \right\}$. It now follows that $z \in E$ and $|x-z| < \delta_{n_0}$ imply $|f(x)-f(z)| < \epsilon$ and thereby $f \in T_\alpha[x]$.

We combine the theorems of Sections 1 and 2 with the above theorem to derive the next two.

3.2. <u>Theorem</u>. Let α be a relation on $\mathcal{P}(\mathbb{R})$ satisfying the conditions A1, A5, A7 and U. Then the following statements are equivalent.

(1) $f \in T_\alpha[x]$ for all $x \in \mathbb{R}$.

(2) $f^{-1}[G] \alpha f^{-1}[G]$ for each open set G .

(3) $f \in \text{comp } T_\alpha$.

3.3. <u>Theorem</u>. Let α be a relation on $\mathcal{P}(\mathbb{R})$ satisfying the conditions A1, A5, A6 and A7. Then the following statements are equivalent.

(1) $f \in T_\alpha[x]$ for all $x \in \mathbb{R}$.

(2) $f^{-1}[G] \alpha f^{-1}[G]$ for each open set G .

(3) $f \in \text{comp}\ \mathcal{T}_\alpha$.

We remark that the conditions A2 and A4 were never used. In Agronsky's Theorem, $f \in \mathcal{B}_1$. This condition does not appear in the above theorems. See 4.6 below for more on the condition $f \in \mathcal{B}_1$.

There are several examples related to derivatives given in [1] by Agronsky. We conclude this section with an example generated by topologies on $\mathbb{R}$. In particular, we consider topologies $\mathcal{T}$ on $\mathbb{R}$ which contain the usual topology on $\mathbb{R}$.

1. The usual topology $\mathcal{E}$ of $\mathbb{R}$.
2. The density topology $\mathcal{Q}$ on $\mathbb{R}$ [2] .
3. The continuity-from-the-right topology $\mathcal{R}$ generated by the half-open intervals $[a,b)$. Of course, continuity-from-the-left will work just as well.
4. The topology $\mathcal{QP}$ generated by the approximately continuous, almost everywhere continuous functions [3], [4], [5].

3.4. Example. Suppose $\mathcal{T}$ is a topology on $\mathbb{R}$ listed above. Let α be defined by $E \alpha F$ if and only if there is $V \in \mathcal{T}$ such that $E \subset V \subset F$.

For these examples we have $E \alpha E$ if and only if $E \in \mathcal{T}$. Clearly, α satisfies the conditions A1, A5 and U. Only the condition A7 requires verification. We verify each case.

1. The usual topology $\mathcal{E}$. Let $E_{n+1} \subset E_n$, $E_n \in \mathcal{E}$ and $\bigcap_{n=1}^{\infty} E_n = \{x\}$. There will be no loss in generality by assuming E_n is an open interval (a_n,b_n) . Let $2\delta_n = \min \left\{ x-a_{n+1},\ b_{n+1}-x \right\}$. Then δ_n decreases to 0 and

$$\{x\} \cup \bigcup_{n=1}^{\infty} \left\{ z \in (a_n, b_n) \mid |x-z| > \delta_n \right\} = (a_1, b_1).$$

2. <u>The density topology</u> $\mathcal{a}$. Let $E_{n+1} \subset E_n$, $E_n \in \mathcal{a}$ and $\bigcap_{n=1}^{\infty} E_n = \{x\}$. Then x is a point of density one for each E_n. Hence there is a sequence δ_n decreasing to zero such that for any interval I with $x \in I$ and $\mu(I) < \delta_n$ we have $(1-2^{-n})\mu(I) \leqslant \mu(E_n \cap I)$. Let $b_n \in (x+\delta_{n+1}, x+\delta_n]$. One easily calculates that

$$(1-2^{-n})\mu([x+\delta_{n+1}, b_n]) \leqslant \mu(E_n \cap [x+\delta_{n+1}, b_n]) + 2^{-n}\delta_{n+1} .$$

Thus, if $[x,b]$ and j are such that $b \in (x+\delta_{j+1}, x+\delta_j]$, we have by repeated use of the above inequality,

$$(1-2^{-j})\ \mu([x,b]) \leqslant$$

$$\leqslant \mu(E_j \cap [x+\delta_{j+1}, b]) + \sum_{k>j} \mu(E_k \cap [x+\delta_{k+1}, x+\delta_k]) + 2^{-j+1}\mu([x,b]).$$

From this inequality we infer that x is a point of density one of the measurable set

$$E = \left(\{x\} \cup \bigcup_{n=1}^{\infty} \{z \in E_n \mid |x-z| > \delta_{n+1}\}\right).$$

It is now easily shown that each point of E is a point of density one. Hence $E \in \mathcal{a}$. We have shown that $\{x\} \propto E$.

3. The continuity-from-the-right topology $\mathcal{R}$. Let $E_{n+1} \subset E_n$, $E_n \in \mathcal{R}$ and $\bigcap_{n=1}^{\infty} E_n = \{x\}$. There will be no loss in generality by assuming $E_n = [x, b_n)$. The remainder of the proof is similar to that of 1 above.

4. The topology $\mathcal{AP}$. We remark that $E \in \mathcal{AP}$ if and only if $E \in \mathcal{A}$ and $E = U \cup Z$ where $U \in \mathcal{E}$ and $\mu(Z) = 0$. Let $E_{n+1} \subset E_n$, $E_n \in \mathcal{A}$ and $\bigcap_{n=1}^{\infty} E_n = \{x\}$. We may assume without loss of generality that $E_n = U_n \cup \{x\}$ with $E_n \in \mathcal{A}$ and $U_n \in \mathcal{E}$. Since $E_{n+1} \subset E_n$, either $x \in U_n$ for all n or $x \notin U_n$ for all but a finite set of indices n. The first case follows from 1 above. The second case follows from 2 above.

4. The necessity of A1-A7 in Agronsky's Theorem.

We investigate the necessity of the conditions A1-A7 in Agronsky's Theorem. It should be said at the outset that these conditions are natural in the study of derivatives as shown by Agronsky. We show that some of these conditions are redundant and that some are not necessary conditions. Only the conditions A4 and A7 appear to be necessary.

4.1. Condition A1. Let α_1 be the relation on $\mathcal{P}(\mathbb{R})$ given by $E \, \alpha_1 \, F$ if and only if $E \subset F$ and F is open. Clearly condition A1 fails and the conditions A2-A6 are easily verified. Note that $E \, \alpha_1 \, E$ if and only if E is open. The condition A7 is verified by using the examples of Section 3 above.

Obviously, $f \in \mathcal{T}_{\alpha_1}[x]$ for all $x \in \mathbb{R}$ if and only if f is continuous. Finally, T_{α_1} is the set of continuous functions. Thus we have shown that the condition A1 is not necessary for Agronsky's Theorem.

4.2. Condition A2. We have seen that condition A2 is redundant among the conditions A1-A7. (Claim 2.6.)

4.3. Condition A3. We have seen that condition A3 is redundant among the conditions A1-A7. (Claim 2.5.)

4.4. Condition A4. We prove the following claim.

Claim. Suppose α is a relation on $\mathcal{P}(\mathbb{R})$ satisfying the conditions A1, A5 and A6 and either $T_{\alpha} \neq 0$ or $T_{\alpha}[x] \neq 0$ for all $x \in \mathbb{R}$. Then α satisfies the condition A4.

Proof. If $\mathbb{R} \,\alpha\, \mathbb{R}$ then condition A5 implies condition A4. Suppose $\mathbb{R}$ is not α-related to itself. Then condition A6 gives the existence of $x_0 \in \mathbb{R}$ such that $\{x_0\}$ is not α-related to $\mathbb{R}$. If $x_0 \in F$ then conditions A1 and A6 imply F is not α-related to itself. We now infer that $T_{\alpha} = 0$ and $T_{\alpha}[x_0] = 0$.

Thus we find that if all of the conditions A1 through A7 except possibly A4 are satisfied and the conclusion of Agronsky's Theorem is true for some f then the condition A4 is necessary.

4.5. Condition A5. Let α_5 be the relation on $\mathcal{P}(\mathbb{R})$ defined by $E \,\alpha_5\, F$ if and only if for each $x \in E$ there is an open set U_x such that either $x \in U_x \subset F$ or $x \in U_x \cup \mathbb{Q} \subset F$, where $\mathbb{Q}$ is the set of rational numbers. One easily verifies that $E \,\alpha_5\, E$ if and only if E is an open set or E

is an open set union $\mathbb{Q}$. Consequently, $\mathbb{Q}\,\alpha_5\,\mathbb{Q}$. Thus, the condition A5 does not hold. It is also easy to verify that the conditions A1, A2, A3, A4, A6 and A7 hold true for α_5.

We show that $f \in \mathcal{T}_{\alpha_5}[x]$ for all $x \in \mathbb{R}$ implies f is continuous. Let x be fixed. Then there is a set E such that $x \in E$, $E\,\alpha_5\,E$ and $f|E$ is continuous at x. If $x \notin \mathbb{Q}$ then x is an interior point of E and hence f is continuous at x. If $x \in \mathbb{Q}$ then $f|\mathbb{Q}$ is continuous at x. It is now an easy exercise to show that a function f is continuous if it is continuous at each $x \notin \mathbb{Q}$ and if $f|\mathbb{Q}$ is continuous.

We finally show that $\mathcal{T}_{\alpha_5}$ is the set of continuous functions. First observe that $\mathbb{Q} \setminus E \neq 0$ and $E\,\alpha_5\,E$ imply E is open. Let $f \in \mathcal{T}_{\alpha_5}$. Suppose $q_1, q_2 \in \mathbb{Q}$ and $f(q_1) < f(q_2)$. Then, for $f(q_1) \leqslant a < f(q_2)$ we have $E_a(f)$ is open. Similarly, for $f(q_1) < a \leqslant f(q_2)$ we have $E^a(f)$ is open. Moreover, if $a \leqslant f(q)$ for all $q \in \mathbb{Q}$ then $E^a(f) = 0$. Similarly, if $a \geqslant f(q)$ for all $q \in \mathbb{Q}$ then $E_a(f) = 0$. We now infer that $E_a(f)$ and $E^a(f)$ are open for all $a \in \mathbb{R}$. This implies f is continuous. Conversely, that each continuous function is a member of $\mathcal{T}_{\alpha_5}$ is easily seen.

The example α_5 shows that condition A5 is not a necessary one in Agronsky's Theorem.

4.6. <u>Condition</u> <u>A6</u>. Let α_6 be the relation on $\mathcal{P}(\mathbb{R})$ defined by $E\,\alpha_6\,F$ if and only if there is a σ-perfect set W such that $E \subset W \subset F$. Clearly, $E\,\alpha_6\,E$ if and only if E is σ-perfect. Since the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is not σ-perfect, we have $\mathbb{R} \setminus \mathbb{Q}$ is not

α_6-related to itself. But, for each $x \in \mathbb{R} \setminus \mathbb{Q}$ we have $\{x\} \ \alpha_6 \ \mathbb{R} \setminus \mathbb{Q}$. Hence condition A6 fails for α_6 . One easily verifies that the conditions A1–A5 and A7 hold true.

From Theorem 3.1, we infer that $f^{-1}[G] \ \alpha_6 \ f^{-1}[G]$ for each open set G implies $f \in T_{\alpha_6}[x]$ for each $x \in \mathbb{R}$. We prove the converse implication under the added hypothesis that $f \in \mathcal{B}_1$. Suppose $f \in \mathcal{B}_1$, $f \in T_{\alpha_6}[x]$ for all $x \in \mathbb{R}$ and G is an open set. Then $f^{-1}[G]$ is an F_σ set. Consequently, $f^{-1}[G]$ is the union of a σ-perfect set A and a countable set B . For each $x \in B$ there is a set E_x and an open set V_x such that $E_x \ \alpha_6 \ E_x$ and $x \in E_x \cap V_x \subset f^{-1}[G]$. Consequently, $f^{-1}[G] = \left(\bigcup_{x \in B} E_x \cap V_x \right) \cup A$. Since $E_x \cap V_x$ is σ-perfect, we have $f^{-1}[G] \ \alpha_6 \ f^{-1}[G]$.

We remark that there is a Baire class 2 function f such that $f \notin T_{\alpha_6}$ and $f \in T_{\alpha_6}[x]$ for all $x \in \mathbb{R}$.

The example α_6 shows that condition A6 of Agronsky's Theorem is not a necessary one.

4.7. <u>Condition A7</u>. The condition A7 is used in a very important way in the proof of Theorem 3.1 above. It is much stronger than necessary in view of Theorem 2.3. The question of the necessity of this condition for Agronsky's Theorem is still open.

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