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Darboux-like Properties of Generalized Derivatives

It has been known for a long time that if f is approximately differentiable, then f'ap has the Darboux property.

In 1976, O'Malley [3] showed that, for every real number λ, if $\{x: f'ap(x) = \lambda\} \neq \emptyset$, then $\{x: f'ap(x) = \lambda\} \cap \{x: f'(x) \text{ exists}\} \neq \emptyset$.

In 1977, O'Malley and Weil [4] showed that if the approximate derivative f'ap of f exists on an interval I_0 and f'ap attains both values M and -M on I_0, then there exists an interval $I \subseteq I_0$ such that f'ap = f' on I and f' attains both values M and -M on I.

Bruckner, O'Malley and Thomson [2] subsequently worked together in introducing the concept of a path derivative as a unifying approach to the study of a number of generalized derivatives. In their paper, they show that most of the familiar properties of the approximate derivative and approximately differentiable functions follow in this setting from two main conditions on the collection E which relate to the "thickness" of the sets E_x and the way they intersect.

Bruckner, O'Malley and Thomson also show that if E is bilateral and has the intersection condition, f is E-differentiable, and f'_E in Baire class 1, then f'_E has the Darboux Property. Upon adding the condition that E be non-porous, they were furthermore able to obtain the same result as that obtained in the M, -M theorem. When replacing both the intersection condition and f'_E in Baire class 1 with the external intersection condition, we can also show the M, -M result.

Next, dropping all intersection conditions, we look for conditions on the system of paths E, the primitive f and f'_E which will give us the same results achieved in the opening three theorems of this article.

If f is continuous, E-differentiable, and E bilateral, then f'_E has the Darboux property. Using Bruckner's Reduction Theorem [1], we can replace f continuous with $f \in [VBG]$.* However, f being Darboux and Baire 1, or Darboux, Baire 2 and VBG does not guarantee that f'_E will be Darboux.

Furthermore, assuming E is nonporous, f a continuous E-differentiable function, and f'_E in Baire class 1, we have O'Malley's Theorem. By using Bruckner's Reduction Theorem and replacing f continuous with $f \in [VBG]$ as done above, we can - again - show O'Malley's Theorem.

Finally, assuming f is continuous and E-differentiable, E is non-porous and f'_E in Baire class 1, we have the M, $-M$ result. Some applications that follow from this are:

1) f'_E has the Darboux property on $\{x: f'(x) \text{ exists}\}$

2) Let I be an interval in $\mathbb{R}$. Let $\Delta(I)$ denote the family of differentiable functions on I, $\Delta_E(I)$ denote the family of E-differentiable functions on I, $N(I)$ denote the family of non-decreasing functions on I, and $P(I)$ be a family of functions on I. If $\Delta(I) \cap P(I) \subseteq N(I)$ for all $I \subseteq \mathbb{R}$, then $\Delta_E(I) \cap P(I) \subseteq N(I)$ for all $I \subseteq \mathbb{R}$.

2a) If $F'_E(x) \geqq 0$ for almost every x at which F is differentiable, then F is non-decreasing.

2b) If F is non-decreasing on every interval on which F is also differentiable, then F is non-decreasing.

3) Let f be continuous and congruently differentiable, and let Q be non-porous at 0. If f'_Q attains both values M and $-M$ on an interval

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*Definition: $f \in [VBG]$ on an interval I iff $I = \bigcup_{n=1}^{\infty} A_n$ such that f is of bounded variation on each A_n and A_n is closed for all n.

I_o then there exists a sub-interval I, such that $f'_Q = f'$ on I, and f' attains both values M and $-M$ on I.

[1] A.M. Bruckner, Lecture Notes in Mathematics: Differentiation of Real Functions, Berlin Heidelburg New York, Springer-Verlag, 1978.

[2] A.M. Bruckner, R.J. O'Malley and B. Thomson, "Path Derivatives : A Unified View of Certain Generalized Derivatives", to appear.

[3] R.J. O'Malley, "The Set where an Approximate Derivative is a Derivative", Proc. Amer. Math. Soc., Volume 54, January 1976.

[4] R.J. O'Malley and C.F. Weil, "The Oscillatory Behavior of Certain Derivatives", Trans. Amer. Math. Soc. Vol. 234 Number 2, 1977.