Krzysztof Ostaszewski, University of Washington, Department of Mathematics, GN-50, Seattle, WA 98195

Density topology and the Luzin (N) condition

This paper is devoted to the relationship between the Luzin (N) condition and continuity in the density topology. A collection of all known facts is presented, then some new observations are made. In particular, it turns out that although the direct relationship between those notions is not found, for a D-continuous function the inverse images of nullsets are nullsets, unless the function has horizontal level sets of positive measure.

The following notation is used:

$|E|$ - the outer Lebesgue measure of a set $E \subset \mathbb{R}$;

$\overline{d}(E, x)$, $\underline{d}(E, x)$, $d(E, x)$ - the upper, lower and ordinary (respectively) density of a set E at a point x.

1.1. A function $f : [a, b] \to \mathbb{R}$ satisfies the **Luzin (N) condition** if $|f(Z)| = 0$ whenever Z is of measure zero.

The **density topology** on the real line consists of all measurable E such that $d(E, x) = 1$ for all $x \in E$.

A function $f : [a, b] \to \mathbb{R}$ is **D-continuous** if it is continuous with respect to the density topology on the domain and the range.

In [5] the following question is posed: does D-continuity imply the Luzin (N) condition? The question is already answered (negatively). It is our intention to discuss it.

1.2. A counterexample, based on the construction of the Peano curve, given in [2], shows that a function which is continuous in the density topology need not satisfy the Luzin (N) condition.

Yet if we assume that $f : [a, b] \to \mathbb{R}$ is D-continuous and one-to-one, then as shown in [3], both f and f^{-1} map sets of measure zero onto sets of measure zero.

In [1] another interesting thing is noted:

Theorem. **A homeomorphism h (in the sense of natural topology) is D-continuous a.e. if and only if h^{-1} is absolutely continuous. Since a homeomorphism is necessarily continuous and increasing, this shows that the statements:**

(i) h is D-continuous a.e.;

(ii) h^{-1} satisfies the Luzin (N) condition;

are equivalent.

One might therefore suspect that D-continuity has something to do with taking sets of measure zero from the range back into the domain as sets of measure zero. In fact, it is so.

1.3. Proposition **Suppose $f : [a, b] \to \mathbb{R}$ is D-continuous. Suppose furthermore that $f^{-1}(\{y\})$ is of measure zero for each $y \in \mathbb{R}$. Then $|f^{-1}(Z)| = 0$ for each Z of measure zero.**

Proof. Since $|f^{-1}(\{y\})| = 0$ for each y, f preserves the upper outer density (see [3]). Suppose $|Z| = 0$ and $|f^{-1}(Z)| > 0$. Then there is an $x \in f^{-1}(Z)$ such that $\overline{d}(f^{-1}(Z), x) > 0$. Yet $d(f(f^{-1}(Z)), f(x)) = 0$, a contradiction.

1.4. Proposition. **Let $f : [a, b] \to \mathbb{R}$. Suppose there exists a surjective function $g : f([a, b]) \to [a, b]$ such that:**

(i) g is D-continuous a.e.;

(ii) $g \circ f$ is an identity functon,

then f satisfies the Luzin (N) condition.

Proof. Suppose $|Z| = 0$ and $|f(Z)| > 0$. Let $y \in f(Z)$ be such that $\overline{d}(f(Z), y) > 0$ and g is D-continuous at y. Write $g(y) = x$. Then $g(y) = x$ and $x \notin g(f(Z) - \{y\})$, so that $\overline{d}(g(f(Z) - \{y\}), x) > 0$, since g is D-continuous at y. But $g(f(Z) - \{y\}) \subset Z$ and $d(Z, x) = 0$, a contradiction.

1.5. Observation. **Let $f : [a, b] \to \mathbb{R}$ be D-continuous. Define**

$\mathrm{F} = \{\alpha \in \mathbb{R} : |f^{-1}(\{\alpha\})| > 0\}$.

Then F is at most countable. Set

$F = \bigcup_{\alpha \in \mathrm{F}} f^{-1}(\{\alpha\}) = f^{-1}(\mathrm{F})$.

The set F is closed in the density topology.

Our observation is that f satisfies the Luzin (N) condition if and only if it satisfies it on the complement of F.

2.1. Proposition. **For a set $E \subset \mathbb{R}$ the following are equivalent:**

(i) E is of measure zero;

(ii) E is nowhere dense in the density topology;

(iii) E is of the first category in the density topology;

(iv) E is closed and discrete in the density topology;

Proof. See [4].

2.2. By the above proposition an arbitrary function satisfies the Luzin (N) condition if it maps sets from any of the four given classes onto sets from another of them. It fact, it suffices to have images of members of those classes being D-Borel (i.e., Lebesgue measurable).

2.3. One can also observe the following two facts:

(i) if f has the property

for any $A \subset I$, $x \in I$, $\overline{d}(A, x) = 0$ implies $\overline{d}(f(A), f(x)) = 0$;

then f satisfies the Luzin (N) condition.

(ii) f satisfies it also if and only if

$\{x \in Z : \overline{d}(Z, x) > 0\} = \phi$ implies

$\{x \in f(Z) : \overline{d}(f(Z), y) > 0\} = \phi$.

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