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SYMMETRIC AND SMOOTH FUNCTIONS: A FEW QUESTIONS AND FEWER ANSWERS

The goal of the talk having the above title, given at the Santa Barbara Symposium, was to summarize some of the efforts to shed light on the continuity and differentiability properties of symmetric and smooth functions and to highlight several remaining questions. The reference list at the end of this summary will furnish an historical background for those wishing to learn more of this topic. Here, I shall simply retell enough of the story to frame the questions posed at Santa Barbara.

All functions considered here are measurable. We let

$$\Delta^2 f(x,h) = f(x+h) + f(x-h) - 2f(x)$$

and say that f is symmetric [smooth] at x if $\Delta^2 f(x,h) = o(1)$ $[o(h)]$ as $h \to 0$. We say that f is symmetric [smooth] if it is symmetric [smooth] at each x in the real line.

If f is symmetric, the set of points of discontinuity of f was shown to be of measure zero and first category by Neugebauer [8]. He showed that a symmetric function is in Baire class one. If f is smooth, then Neugebauer [8,9] showed that the set of discontinuities is nowhere dense and countable. O'Malley [11] subsequently observed that a smooth function must belong to class Baire* one. Recently, Larson and I [3] showed that the set of discontinuities of a smooth function can be characterized as a scattered (clairsemé) set. Thus, the first problem to be posed here is to

1. Characterize the set of points of discontinuity of a symmetric function.

Two questions which may shed some light on 1. are

1a. Must the set of points of discontinuity of a symmetric function be countable?

1b. Must the set of points of discontinuity of a symmetric function be σ-porous?

The notions of approximately symmetric, approximately smooth, L_p symmetric, and L_p smooth functions ($p \geq 1$) have been defined by Neugebauer [9] in a natural manner. He showed that the set of points of discontinuity of an L_p smooth function is nowhere dense and that any closed nowhere dense set can serve as such a set of discontinuity points. He further showed that L_p symmetric functions are in Baire class one and O'Malley [11] showed that L_p smooth functions are in Baire* one. Neugebauer [9] raised the natural question as to whether or not the set of points where an L_p smooth function fails to be L_p continuous must be countable. Humke and I [2] have answered this in the negative by constructing an L_p smooth function which fails to be even approximately continuous at uncountably many points.

2. Characterize the set of points at which an L_p smooth function is L_p discontinuous. (Must it be σ-porous?)

3. Characterize the set of points at which an L_p symmetric function is L_p discontinuous. (Must it be σ-porous?)

Let's let problems 4. and 5. denote the approximate versions of 2. and 3. and not list them separately so that the cursory reader will be perplexed as to where they are. We should note that approximate symmetric functions are in Baire class one [6]. However, our attempts to show that approximately smooth functions must be in Baire* one have been frustrating. Larson [7] has successfully shown that an approximately continuous approximately smooth function is in Baire* one.

6. Is an approximately smooth function in Baire* one?

To this point not much has been said about the differentiability of smooth functions. However, much is known and the results are interesting. In 1964 Neugebauer collected and improved the "state of the art" by showing that 1) a smooth function is differentiable on a set having the power of the continuum in each interval and 2) if f is continuous then f' has the Darboux property on this set. Capitalizing on the fact that smooth functions are in Baire* one, O'Malley [11] showed that the continuity condition in 2) can be replaced by the Darboux property assumption for f. In the same manner Neugebauer followed by O'Malley obtained the analogous result for L_p smooth functions; i.e. an L_p smooth function has an L_p derivative at each point of a set having the power of the continuum in every interval and if f has the Darboux property, then the L_p derivative of f has the Darboux property on its set of existence. Because we are stymied by problem 6 above, the best result known so far in the approximate case is that an approximately continuous, approximately smooth function will possess an approximate derivative at each point of a set haivng the power of the continuum in every interval and f'_{ap} has the Darboux property on this set. This result is obtained by following the Neugebauer O'Malley path used above in both the usual and L_p cases or an alternate proof was given in [10].

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