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On generalizations of exact Peano derivatives and integrals

- via the coefficient problems of convergent trigonometric series -

1. Introduction

For a given trigonometric series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (T)$$

consider the following conditions: (k being a positive integer)

(C_0) (T) converges to $f_0(x)$ for every x in R = $(-\infty,+\infty)$;

(C_k) (T) is (C,k)-summable to $f_k(x)$ for every x in R;

(C_∞) (T) converges in the distributional sense to a distribution f_∞ in R.

For α = 0, k, or ∞, the (C_α)-coefficient problem is: how can one represent the coefficients a_n, b_n in terms of the "sum" f_α of (T)? We will discuss some solutions to these problems and indicate how certain integrals used in the solutions are related to the concept of exact Peano derivatives. Furthermore, a generalization of the later concept arising naturally through the discussion will be considered.

Before discussing the solutions, first let us recall that if the series (T) converges to f(x) uniformly for x in R, then the coefficients a_n and b_n

can be represented by the Euler-Fourier formulae:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt,$$

where the integral involved is the definite Riemann integral. Therefore, to solve the (C_α)-coefficient problem, one essentially needs to extend the definite Riemann integral so that modified Euler-Fourier formulae would make sense even if the continuous function f is replaced by the "sum" f_α of the series (T).

2. Solutions to the (C_0)-coefficient problem.

It is well-known that if two trigonometric series [of the form (T)] converge pointwisely to the same function, then the sequences of the coefficients of the two series are identical. Hence the (C_0)-coefficient problem is well-posed in the sense that if the condition (C_0) holds then the coefficients a_n, b_n are completely determined by the sum function f_0.

Before we discuss the solutions to the problem, we note that the Lebesgue integral or even the Denjoy integral in the wide sense (cf. Saks [11]) is not good enough to solve the coefficient problem. For, there are everywhere convergent trigonometric series of which the sum functions are not even Denjoy integrable. For example, (T) is such a series if $a_n = 0$ and $b_n = 1/\ell n(n+1)$.

The (C_0)-coefficient problem has been solved by Denjoy [2], Marcinkiewicz and Zygmund [9], James [3], Burkill [1], respectively. They have developed a theory of the totalization symétrique à deux degrés (to be called the $D_{2,s}$-integral), the (T)-integral, the P^2-integral, and the SCP-integral, respectively. If A denotes one of the integral, then the (C_0)-coefficient problem is solved in the following sense: if (C_0) holds then the functions $f_0(x)\cos nx$ and $f_0(x)\sin nx$ are A-integrable and the coefficients a_n and b_n are given by some modified Euler-Fourier formulae using the <u>definite</u> A-integral.

To indicate what these integrals are, let us recall that the Denjoy-Perron integral (cf. [11]) solves the classical primitive problem in the sense that it integrates every derivative and recpature its primitive up to a constant term. The $D_{2,s}$-integral and the P^2-integral have been developed in a similar manner that essentially each of these integrals integrates the second symmetric derivative D^2F and recpatures the second primitive F up to a <u>linear</u> term, where D^2F is defined by $D^2F(x) = \lim_{h\to 0}[F(x+h)+F(x-h)-2F(x)]/2h^2$.

Due to the nature of being second order, the definite integral of a function is related to the second variation of its second primitive. Thus, the modified Euler-Fourier formulae using these integrals in solving the (C_0)-coefficient problem look like

$$a_n = \frac{1}{\pi^2} \int_{(-2\pi,0,2\pi)} f(t)\cos nt\,dt \quad ,$$

where the definite integral essentially means the second variation of the second primitive of the function $f(t)\cos nt$ at the three points $-2\pi, 0, 2\pi$.

On the other hand, the (T)-integral and the SCP-integral are first order in nature. Each of these integrals is developed so that essentially it integrates the second symmetric derivative D^2F and recaptures the ordinary derivative F' almost everywhere and up to a constant term. Using these integrals, the modified Euler-Fourier formulae look like

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(t)\cos nt\,dt \quad ,$$

where α can be any point in a set of full measure, and the definite integral essentially means the first variation of the first derivative of the second primitive of the function $f(t)\cos nt$ at the points α and $\alpha + 2\pi$, i.e. if $D^2F(t) = f(t)\cos nt$, then $a_n = F'(\alpha+2\pi) - F'(\alpha)$. This sounds more like the fundamental theorem in integral calculus. Although the indefinite integral in such a theory is only almost everywhere defined, it must be nice in the sense that it is almost everywhere an exact derivative. Extending the the last viewpoint, we are going to see that the exact $(k+1)^{th}$ Peano derivative plays a role in a certain integral theory used in solving the (C_k)-coefficient problem. (Cf. next section.)

3. Solutions to (C_k)-coefficient problems.

The (C_k)-coefficient problem is not well-posed in general. For example, the series $\sum_{n=1}^{\infty} n\sin nx$ is (C,2)-summable to 0 but the coefficients are not zero. However, if some extra conditions are imposed on the coefficients a_n, b_n, then one still can consider the problem of representing a_n and b_n in terms of the sum function f_k. Such problems have been investigated by James [4]. He has extended his P^2-integral to a P^{k+2}-integral which in essence integrates the (k+2)-symmetric derivative $D^{k+2}F$ and recaptures the $(k+2)^{th}$ primitive F up to a polynomial of degree k+1, and hence the modified Euler-Fourier formulae are much more complicated and will not be reviewed here. On the other hand, extending the SCP-integral, a G_{k+1}-integral has been obtained in [8], which has been used to solve the same (C_k)-coefficient problem considered by James. The G_{k+1}-integral is still of order 1 in nature. In essence, it integrates the $(k+2)^{th}$ symmetric derivative $D^{k+2}F$ and recaptures the $(k+1)^{th}$ Peano derivative $F_{(k+1)}$ almost everywhere and up to a constant term. The Euler-Fourier formulae in the solution for the (C_k)-coefficient problem considered by James using the G_{k+1}-integral look like

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f_k(t)\cos nt\,dt = F_{(k+1)}(\alpha+2\pi) - F_{(k+1)}(\alpha) \quad ,$$

where F is a function such that $D^{k+2}F(t) = f_k(t)\cos nt$ subject to some extra conditions. An indefinite G_{k+1} integral is only almost everywhere defined, but must be almost everywhere an exact $(k+1)^{th}$ Peano derivative.

4. A perfect solution for the (C_∞)-coefficient problem.

What does $\int_{-\pi}^{\pi} f_\infty(t)\cos nt\,dt$ mean?

Unlike the case for the point functions (i.e. real-valued functions of a real variable), every distribution f is the distributional derivative of a distribution L. Hence, to solve the (C_∞)-coefficient problem, we first try to see what the <u>definite</u> integral $\int_a^b f(t)dt$ means in terms of the distribution L. The concept of the value of a distribution T at a point x_0 in R is the key. We recall it below.

A distribution T is said to have a value at x_0 if the distribution $T(ax+x_0)$ converges in the distributional sense as $a \to 0$, i.e. for each test function ϕ,

$$\int T(ax+x_0)\phi(x)dx \equiv \int T(x)\frac{1}{|a|}\phi\left(\frac{x-x_0}{a}\right)dx$$

converges as $a \to 0$. One shows that if T has value at x_0, then $\lim_{a\to 0} T(ax+x_0)$ is a constant distribution, and this constant is called the value of T at x_0, and will be denoted as $v(T,x_0)$.

Given a distribution f and two reals a,b with $a \neq b$, define

$$T(x) \equiv \int_a^b f(x+t)dt \equiv L(x+b) - L(x+a),$$

where L is a distribution with $L' = f$. (Note that it is well-defined since two primitives of the same distribution can be different by at most a constant distribution.) If T has value at 0, then define

$$\int_a^b f(t)dt = v(T,0),$$

which is called the definite integral of the distribution f from a to b.

It is easily shown that if f is a periodic distribution (with period 2π), then the definite integral $\int_{-\pi}^{\pi} f(t)dt$ exists. Thus, for every periodic distribution f, the Fourier coefficients of f, (i.e. $\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt dt$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt dt$) make sense, and we have the Fourier series for such a distribution. Furthermore, we have the following "perfect" situation (cf. [10]):

<u>Theorem 1</u>. If f is a periodic distribution (with period 2π), then its Fourier series converges in the distributional sense to f. Conversely, if the trigonometric series (T) converges in the distributional sense to f_∞, then (T) is the Fourier series of f_∞.

Thus, in the distribution theory, convergent trigonometric series and Fourier series of periodic distributions are identical. This is beautiful. But what does it imply if we are going to stay within the domain of the classical theory of point functions? We will discuss one implication in the next section.

5. Tauberian conditions and exact generalized Peano derivatives.

It is well-known that if (C_0) holds then so does (C_k) (and $f_k=f_0$). It is also easy to show that if (C_k) holds, then so does (C_∞). (But in this case the relation between the function f_k and the distribution f_∞ requires further investigation and will not be discussed here.) The converse of each of the above results is not true. However, (C_k) plus a certain (Tauberian) condition would imply (C_0). This kind of results are also well-known. On the other hand, what kind of conditions plus (C_∞) would imply (C_k) for some k seems to be left unnoticed. We give such a Tauberian condition in the following result (cf. [13]):

Theorem 2. If (C_∞) holds and if f_∞ has value at x_0, then there exists a positive integer k such that the series (T) is (C,k)-summable to $v(f_\infty,x_0)$ at $x = x_0$.

With the Tauberian condition in theorem 2, we are led to consider the following class of point functions:

let $V = \{f \mid$ there exists a distribution T of finite order such that $v(T,x)$ exists and is equal to $f(x)$ for all $x\}$.

How do we characterize V within the domain of point functions (i.e. without using the concept of distributions)? It happens that a sightly generalized notion of the exact Peano derivative will do the job. In fact, we have the following result [cf. [7]):

Theorem 3. A function f is in V if and only if there exist a continuous function F and a positive integer k such that for each x in R there exists a positive integer $n = n(x)$ such that $G_{(n+k)}(x) = f(x)$, where G is a

n^{th} primitive of F.

Here, $G_{(n+k)}(x)$ denotes the $(n+k)^{th}$ Peano derivative of G at x. Recall that a function f is called an exact Peano derivative of order k if there exists a function F such that $F_{(k)}(x) = f(x)$ for all x. From theorem 3, we see that for every k, an exact Peano derivative of order k is in V. However, there are functions in V which are not exact Peano derivatives of any (finite) order (cf. [5]). (Such functions may as well called the exact Peano derivatives of ∞ order.)

Keeping in mind the perfect result (i.e. theorem 1) in the distribution theory, we are hoping that the class V (of the exact generalized Peano derivatives) could be utilized to improve and/or unify some results in the theory of point functions.

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