A SIMPLE PROOF OF THE INTEGRATION BY PARTS THEOREM FOR PERRON INTEGRALS

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(In this note the P^*-, RS-, P^*_{ap}- integrals denote the Perron, Riemann-Stieltjes and approximately continuous Perron integral of Burkill respectively).

1. Theorem, [3]. If $f \in P^*(a,b)$, $F = P^*\text{-}\int f$, G of bounded variation then $fG \in P^*(a,b)$ and

$$P^* - \int_a^b fG = F(b)G(b) - F(a)G(a) - RS - \int_a^b FdG.$$

It is sufficient to prove this for G increasing, bounded, with $G(a) = 0$.

A natural candidate for a major function of fG is

$$R(x) = M(x)G(x) - \int_a^x MdG, \quad a \le x \le b, \tag{1}$$

where M is a major function of f; replacing M by m, a minor function of f, we get r, a natural candidate for a minor function of fG.

Unfortunately R is only a left major function of fG; that is, it satisfies the required differential inequalities for left derivatives; only - similarly r is only a left minor function. However if we replace, in (1), G by G^*, where $G^*(x) = G(x) - G(b)$, $a \leq x \leq b$, the function R^* is a right minor function of fG^*. Hence $m(x)G(b) + R^*(x)$ is a right minor function of fG, and similarly $M(x)G(b) + r^*(x)$ is a right major function.

So we can construct left, and right, major and minor functions of fG: this, together with Ridder's observation, [4], that the McShane Perron integral is trivially equivalent to the P^*-integral completes the proof.

2. Theorem, [2]. Let $f \in P^*_{ap}(a,b)$, $f = P^*_{ap}\text{-}\int f$, g of bounded variation, $G = \int g$, and if $F \in P^*(a,b)$ then $fG \in P^*_{ap}(ab)$ and

$$P^*_{ap}\text{-}\int_u^b fG = F(b)G(b) - F(a)G(a) - P^*\text{-}\int_a^b Fg. \qquad (2)$$

(The need to assume $F \in P^*$ is demonstrated in [2]).

In this case the analogous R is a major function and so the proof does not present the difficulties met in the classical case; (this is because here $G' = g$ n.e., whereas above $G' = g$ a.e.) Alternatively we can remark that the right-hand side of (2) is, as a function of b, $[ACG^*_{ap}]$, [1] and has an approximate derivative equal to fG a.e. Another proof is given in [2].

Bibliography

1. P. S. Bullen, The Burkill approximately continuous integral, J. Austral. Math. Soc., (Series A) 35 (1983), 236-253.

2. P. S. Chakrabarti and S. N. Mukhopadhyay, Integration by parts for certain approximate CP-integrals, Bull. Inst. Math. Acad. Sinica, 9(1981), 493-507.

3. L. Gordon and S. Lasher, An elementary proof of integration by parts for the Perron integral, Proc. Amer. Math. Soc., 18(1967), 394-398.

4. J. Ridder, Ueber Definitionen von Perron-Integralen I, Indag. Math., 9 (1947), 227-235.