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The Peano derivative: What's known and what isn't

The Peano derivative is a notion of nth order differentiation which flourished in the first half of the twentieth century attracting the attention not only of real analysts but also that of those working in Fourier series. It has fallen completely out of favor with harmonic analysts and has fared only slightly better among real analysts. The aim of this paper is to attempt to revive some interest in this topic at least among those working in real variables. It is hoped to accomplish this goal by first bringing to the reader's attention several early papers in the area which, in light of later results, become all the more interesting, and then stating several unsolved problems which, if resolved, would clarify the relationship between the Peano derivative and various notions of first order differentiation.

The place to begin is, of course, the conventions, notations and basic definitions. The letter I will denote a fixed, closed subinterval of the real line and for any set E of real numbers $\bar{E}$ will denote the closure of E. The letters m,n and p will denote nonnegative integers. All functions will be real-valued or extended real-valued functions defined on I.

Definition: A function f is said to have an nth Peano derivative at $x \in I$ if there are numbers $f_1(x), f_2(x), \ldots, f_n(x)$ such that for $y \in I$

(1) $f(y) = f(x) + f_1(x)(y-x) + \cdots + [f_n(x) + \varepsilon_n(x,y)](y-x)^n/n!$

where $\lim_{y\to x} \varepsilon_n(x,y) = 0$.

The number $f_n(x)$ is called the nth Peano derivative of f at x.

The classical Taylor Theorem asserts that if f has an ordinatry nth derivative, $f^{(n)}(x)$, at x, then $f_n(x)$ exists and $f_n(x) = f^{(n)}(x)$. The converse is false if $n \geq 2$ as can be seen from the standard example involving $x^{n+1} \sin x^{-1}$. More to the point the nth Peano derivative is more general than the ordinary nth derivative in that the existence of $f^{(n)}(x)$ implies the existence of the lower order derivatives in an entire neighborhood of x, while the existence of $f_n(x)$ implies only the existence of $f_1(x), f_2(x), \ldots, f_{n-1}(x)$. More general yet are the several notions of nth order differentiation defined as limits of various nth order difference quotients. For example, the nth Riemann derivative is

$$\lim_{h\to 0} \Sigma_{j=0}^{n} \frac{(-1)^{n-j}\binom{n}{j} f(x + jh - \frac{1}{2}nh)}{h^n} .$$

The existence of such a limit on a set implies the existence of f_n a.e. on that set. (See [7], Theorem 1, page 2.) However, the existence of f_n on an interval implies the existence of $f^{(n)}$ only on an open, dense set. These facts seem to make a strong case for the assertion that the nth Peano derivative is the best notion of nth order differentiation.

The transition from the nth Peano derivative to the nth Peano derivates and hence to the definition of infinite nth Peano derivatives is as one might expect. Suppose that $f_{n-1}(x)$ exists and set

$$\mathcal{O}_n(f;x,y) = \mathcal{O}_n(x,y) = \frac{f(y) - f(x) - f_1(x)(y-x) - \cdots - f_{n-1}(x)\frac{(y-x)^{n-1}}{(n-1)!}}{(y-x)^n/n!}$$

Then clearly $f_n(x)$ exists if and only if $\lim_{y\to x}\mathcal{O}_n(x,y)$ exists and in that case

(2) $$f_n(x) = \lim_{y\to x}\mathcal{O}_n(x,y).$$

Consequently, it is quite natural to define the nth Peano derivates in terms of $\mathcal{O}_n$.

Definition: The upper, right nth Peano derivate is denoted by $f_n^+(x)$ and is defined by

$$f_n^+(x) = \overline{\lim}_{y\to x^+} \mathcal{O}_n(x,y).$$

The other three derivates, $f_{n+}(x)$, $f_n^-(x)$, and $f_{n-}(x)$ are all defined similarly. Equation (2) can serve as a means to define the equality $f_n(x) = \infty$ or $f_n(x) = -\infty$.

Although this notion of nth order differentiation was introduced much earlier, (in [11] and [14]) the first extensive study of the concept seems to be due to Denjoy [5] in 1935. The foundation of the first part of his work is the so-called La Grange Formula which says how to recover the coefficients of a polynomial of degree n from the values of the polynomial at n+1 distinct points. Assume f_n exists on I and for α and η (small) positive numbers let $E_n(f;\alpha,\eta) = \{x \in I : y \in I$ and $|x-y| < \eta$ implies $|\varepsilon_n(x,y)| \leq \alpha\}$. Now the behavior of $f_1, f_2, \ldots, f_n$ on $\bar{E}_n(f;\alpha,\eta)$ is investigated. Let $z \in \bar{E}_n(f;\alpha,\eta)$ and pick $z_0, z_1, \ldots, z_n$ any n+1 distinct points such that $|z-z_i| < \eta/2$ for each i. Let $x \in E_n(f;\alpha,\eta)$ with $|x-z| < \eta/2$. Then for each $i = 0,1,\ldots,n$, according to (1)

$$f(z_i) = f(x) + f_1(x)(z_i-x)+\cdots+[f_n(x) + \varepsilon_n(x,z_i)](z_i-x)^n/n!$$

Using the La Grange Formula an expression for $f_p(x)$ is obtained in terms of the numbers $f(z_0), f(z_1), \ldots, f(z_n), \varepsilon_n(z,z_0), \varepsilon_n(x,z_1), \ldots, \varepsilon_n(x,z_n)$. Since the first half of these are fixed and the second half are bounded by α for $|z-x| < \eta/2$ and $x \in E_n(f;\alpha,\eta)$, $f_1(x), \ldots, f_n(x)$ are bounded. Pick any f_{p_o}, $p_o \leq n-1$, any limit value, a_{p_o}, of $f_{p_o}(x)$ as x tends to z through $E_n(f;\alpha,\eta)$ and a sequence from $E_n(f;\alpha,\eta)$ on which f_{p_o} tends to a_{p_o}. Considering the remaining functions f_p one at a time choose limiting values and thin out the sequence to arrive at $a_1, \ldots, a_p$ and $\{x_k\}$ in $E_n(f;\alpha,\eta)$ such that $\lim_{k\to\infty} f_p(x_k) = a_p$ for $p = 1,2,\ldots n$. Since f is continuous, $\lim_{k\to\infty} f(x_k) = f(z)$. Now let y be any number with $|y-z| < \eta$. Then for k so large that $|y-x_k| < \eta$

$$f(y) = f(x_k) + f_1(x_k)(y-x_k)+\cdots+[f_n(x_k) + \varepsilon_n(x_k,y)](y-x_k)^n/n!$$

and $|\varepsilon_n(x_k,y)| \leq \alpha$. Letting k tend to ∞ gives

$$f(y) = f(z) + a_1(y-z) +\cdots+[a_n+\varepsilon(y)] \quad (y-z)^n/n!$$

where $|\varepsilon(y)| \leq \alpha$. Since this holds for all such y, for at least $p = 1,\ldots,n-1$ it follows that $f_p(z) = a_p$. Since a_{p_o} was arbitrary, f_{p_0} is continuous at z relative to $E_n(f;\alpha,\eta)$ and since p_0 was arbitrary, the same is true of $f_1,\ldots,f_{n-1}$. The continuity of these functions relative to $\bar{E}_n(f;\alpha,\eta)$ follows at once. The conclusion that can be drawn from the above for $p = n$ is that both the limit superior and limit inferior of $f_n(x)$ as x tends to z through $\bar{E}_n(f;\alpha,\eta)$ differ from $f_n(z)$ by no more than α. Finally, it follows that $|\epsilon_n(z,y)| \leq 2\alpha$ if $|z-y| < \eta$.

The observation about the behavior of f_n on $\bar{E}_n(f;\alpha,\eta)$ together with the fact that $\bigcup_{m=1}^{\infty} E_n(f;\alpha,1/m) = I$ for any α can be used to imply that f_n is a function of Baire class 1. The continuity of $f_1,\ldots,f_{n-1}$ on $\bar{E}_n(f;\alpha,\eta)$ implies that these functions are B_1^* or generalized continuous on I. As was pointed out quite recently by David Preiss this latter fact when combined with (2) proves that f_n is Baire 1 even when f_n is permitted to have infinite values. This answers a question raised in [6]. Lastly, the observation concerning $\epsilon_n(z,y)$ implies $\bar{E}_n(f;\alpha,\eta) \subset E_n(f;2\alpha,\eta)$. Consequently facts about elements of $E_n(f;2\alpha,\eta)$ hold for $\bar{E}_n(f;\alpha,\eta)$ as well.

Such a fact follows from another application of the La Grange Formula. It can be shown that there is a number B_n depending only on n such that if $x,y \in E_n(f;\alpha,\eta)$ with $|x-y| < \eta$, then for $p = 1,\ldots,n$

$$f_p(y) = f_p(x) + f_{p+1}(x)(y-x)+\cdots+[f_n(x) + \alpha\beta_p(x,y)]\frac{(y-x)^{n-p}}{(n-p)!}$$

where $|\beta_p(x,y)| \leq B_n$. Due to the comment at the end of the preceding paragraph, if P is any perfect subset of I, then on a dense, G_δ subset of P and for $p = 1,\ldots,n-1$, f_p

has an (n-p)th Peano derivative relative to P and $(f_p)_m = f_{p+m}$ for $m \leq n-p$. By a similar argument, if f has finite nth Peano derivates on I, then any perfect subset, P, has a dense, G_δ subset on which each f_p, $p = 1,\ldots,n-1$ has an (n-p-1)th Peano derivative and finite (n-p)th Peano derivates all relative to P with the expected equalities.

Perhaps the most difficult theorem in the paper also has to do with nth Peano derivates and is as follows.

Theorem: Suppose f is continuous on I. Then almost everywhere on the set where the nth Peano derivates exist and both from at least one side are finite, f_n exists (and is finite).

To see how this theorem relates to it, the Denjoy-Saks-Young Theorem is stated as follows: If f is an arbitrary function, then (i) almost everywhere on the set where both (first order) derivates from at least one side are finite, f' exists (and is finite), (ii) almost everywhere on the set where all derivates are infinite, the two uppers are $+\infty$ and the two lowere are $-\infty$, and (iii) almost everywhere on the set where from at least one side one derivate is finite and the other infinite, one pair of opposite derivates are finite and equal while the other pair are opposite infinities. (The upper right and lower left derivates are one pair of opposite derivates.) The theorem stated above is the generalization to nth Peano derivates of (i) with the added assumption that f is continuous. An obvious question is if the continuity assumption can be removed. The other obvious question, whether (ii) and/or (iii) hold for nth Peano derivates, was posed by Denjoy in his paper and answered later.

The first negative answer to (ii) was given by Frédéric Roger [13]. He constructed an example of a continuous function for which $f_2^+ = f_{2+} = \infty$ and $f_2^- = f_{2-} = -\infty$ on a set of positive measure. Ernest Corominas [3] presented a second negative answer to (ii) and also answered (iii) again in the negative. Concerning (ii) he not only gave an example like the one due to Roger, but also found a continuous function for which $f_n = \infty$ on a set of positive measure. For (iii) he again discovered two examples; one of a continuous function having, on a set of positive measure, $f_n^+ = \infty$, $f_{n-} = -\infty$ $-\infty < f_n^- < 0 < f_+ < \infty$, the other a continuous function with $f_n^+ = f_n^- = \infty$ and $0 < f_{n+}, f_{n-} < \infty$. These four examples are just a small part of the total contribution to the theory made by Corominas. He published a lengthy work in 1953 which will be discussed next.

Corominas based his study [4] of the nth Peano derivative on the notion of the nth divided difference which is now recalled. For $n = 1$ and $x_0, x_1 \in I$, $x_0 \neq x_1$ let

$$V_1(f;x_0,x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} .$$

Suppose $V_n(f;x_0,\ldots,x_n)$ has been defined when $x_0,\ldots,x_n$ are $n+1$ distinct elements of I. Let $x_0,\ldots,x_{n+1}$ be $n+2$ distince elements of I and set

$$V_{n+1}(f;x_0,\ldots,x_{n+1}) = \frac{V_n(f;x_0,\ldots,x_n) - V_n(f;x_1,\ldots,x_{n+1})}{x_0 - x_{n+1}} .$$

It is possible to prove that

$$V_n(f;x_0,\ldots,x_n) = \Sigma_{i=0}^{n} f(x_i)/\varphi'(x_i)$$

where $\varphi(x) = \Pi_{i=0}^{n}(x-x_i)$ and consequently $V_n(f;x_0,\ldots,x_n)$ is independent of the order in which the numbers $x_0,\ldots,x_n$ are listed. Corominas takes this concept one step farther defining $V_n(f;x_0,\ldots,x_n)$ when the numbers $x_0,\ldots,x_n$ are not necessarily distinct by setting

$$V_n(f;x_0,\ldots,x_n) = \lim_{y_0\to x_0} \cdots \lim_{y_n\to x_n} V_n(f;y_0,\ldots,y_n)$$

when this iterated limit exists. The connection between this iterated limit and the nth Peano derivative is given in the following theorem.

Theorem: The divided difference $V_n(f;x_0,\ldots,x_0)$ exists if and only if $f_n(x_0)$ exists. Moreover, when $f_n(x_0)$ exists $V_n(f;x,x_0,\ldots,x_0) = (f_n(x_0) + \epsilon_n(x_0,x))/n!$ and $V_n(f;x_0,\ldots,x_0) = f_n(x_0)/n!$.

In fact it is not hard to see that if a number, c, appears p times in the finite sequence $x_0,x_1,\ldots,x_n$, then the iterated limit exists if and only if $f_{p-1}(c)$ exists.

The first part of the paper deals with mean value theorems. A few examples follow. Assume throughout that f_n exists and is finite on I.

Theorem: If $x_0,\ldots,x_n$ are $n+1$ distinct points in I, then there is a number c between $\min\{x_i : i = 0,\ldots,n\}$ and $\max\{x_i : i = 0,\ldots,n\}$ such that $V_n(f;x_0,\ldots,x_n) = f_n(c)/n!$. Conversely, if $f_n(c)$ is neither the maximum nor the minimum value of f_n on I, then there are distinct $x_0,\ldots,x_n$ in I such that $V_n(f;x_0,\ldots,x_n) = f_n(c)/n!$

Corollary: The function f_n has the Darboux property on I.

Theorem: If $m < n$ and if $x_0, \ldots, x_{n-m}$ are $n-m+1$ distinct points in I, then there is a number c as before such that $V_{n-m}(f_m; x_0, \ldots, x_{n-m}) = f_n(c)/(n-m)!$

Theorem: If $x_0, \ldots, x_n$ are $n+1$ (not necessarily distinct) points in I, then there is a number c as before such that $V_n(f; x_0, \ldots, x_n) = f_n(c)/n!$.

These mean value theorems are followed by analogous Cauchy-type mean value theorems from which Taylor polynomial-type expansions with La Grange-type remainders are established. More significantly the mean value theorem concerning f_n above is used to prove that if f_n is bounded above or below on I, then $f_n = f^{(n)}$ on I. One might ask if that mean value theorem is true if the assumption of distinct points is dropped. A partial, positive answer was unknowingly furnished in the article by Oliver to be mentioned next.

Since his paper appeared in 1954, Oliver [8] was probably unaware of the work of Corominas, for he established many of the same theorems using different techniques. For example he proved the Darboux property and that bounded above or below implies $f_n = f^{(n)}$. The Darboux property is proved by induction on n along with his mean value theorem which is stated here for the purpose of comparison.

Theorem: If f_n exists (finitely) on I and if $x, y \in I$, then there is a number c between x and y such that

$$\frac{f_m(y) - f_m(x) - f_{m+1}(x)(y-x) - \cdots - f_{n-1}(x)\frac{(y-x)^{n-m-1}}{(n-m-1)!}}{(y-x)^{n-m}/(n-m)!} = f_n(c).$$

Oliver also proved that f_n possesses the Denjoy property or the M_2 property of Zahorski.

As was already noted if infinite values are allowed, f_n is still a function of Baire class 1. In [2] it is shown that $f_n(x) = \lim_{x_n \to x} \cdots \lim_{x_0 \to x} V_n(f; x_0, \ldots, x_n)$ also holds when infinite values are permitted for $f_n(x)$. So it seems reasonable to ask what of the other results established for finite Peano derivatives in the works discussed above also hold when $f_n(x)$ can be ∞ or $-\infty$.

There are several seemingly difficult questions that can be posed for the finite Peano derivative. In 1982, David Preiss [12] characterized the associated sets (the associated sets of a function f are the sets of the form $\{x : f(x) < a\}$ and $\{x : f(x) > a\}$ of finite derivatives. (He also characterized associated sets for derivatives of continuous functions and derivatives of arbitrary functions.) In that paper he showed that the same conditions characterize the associated sets of approximate derivatives. It would be interesting to know if the same conditions characterize the associated sets of finite nth Peano derivatives and also nth Peano derivatives allowing infinite values. The next question relates to the notion of selective derivative introduced by Richard O'Malley [9] in 1977. He showed that every approximate derivative is a selective derivative of its primitive. In addition he has shown [10] that there is a sequence $\{F_n\}$ of closed sets whose union is I such that for each n and each

$x \in F_n$, $f_{ap}'(x) = \lim_{y \to x,\ y \in F_n} (f(y) - f(x))/(y-x)$. In this case it is said that f_{ap}' is a composite derivative of f. The two obvious questions that arise in this context are: (1) is f_n a selective derivative and if so is it a selective derivative of f_{n-1}, and (2) is f_n a composite derivative and if so is it a composite derivative of f_{n-1}? And, finally, in [1] it was shown that every approximate derivative can be written in the form $g' + hk'$ where g, h, and k are differentiable functions. It is reasonable to ask if such a representation is possible for nth Peano derivatives.

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