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# ON THE GENERAL THEORY OF POINT SETS

As elegantly discussed in Oxtoby's book, "Measure and Category," there are far-reaching analogies between the topological concepts of Baire and the measure-theoretic concepts of Lebesgue. These analogies gradually emerged from the work of Lebesgue, Denjoy, Sierpiński, and Marczewski (cf. [8], [9], [1], [2], [22], [23], [26], [29], [25]).

Given any statement $\pi$ containing only set-theoretical concepts and the terms "nowhere dense," "first category," "second category," and "has the Baire property," one obtains a statement

$$\rho = \Phi(\pi)$$

concerning Lebesgue measure, by applying the following substitution matrix to the statement $\pi$

$$\Phi = \begin{bmatrix} \text{nowhere dense} & , & \text{measure zero} \\ \text{first category} & , & \text{measure zero} \\ \text{second category} & , & \text{positive outer measure} \\ \text{has the Baire property} & , & \text{is measurable} \end{bmatrix}$$

For instance, the true statement

($\pi$) A set S is of first category if and only if every subset of S has the Baire property.

is transformed into the true statement

($\rho$) A set S is of measure zero if and only if every subset of S is measurable.

General frameworks for unifying certain of these analogies were considered by Marczewski, Hausdorff, and Kondô (cf. [26], [4], [5], [6]). These attempts, however, had limited success since they were based on secondary properties rather than on primary properties. In this connection, we note that on the real line **R** the concepts of Baire are definable in terms of closed intervals, while the corresponding concepts of Lebesgue are similarly definable in terms of closed sets of positive measure. A much more extensive unification has been achieved by means of the axiomatically defined notion of a category base, which generalizes the notion of a topology (cf. [11] — [20]).

DEFINITION. A pair $(X, \mathcal{C})$, where X is a nonempty set and $\mathcal{C}$ is a family of subsets of X is called a category base if the nonempty sets in $\mathcal{C}$, called regions, satisfy the following axioms:

1. Every point of X belongs to some region; i.e., $X = \bigcup\mathcal{C}$
2. Suppose A is a region and $\mathcal{D}$ is a nonempty family of disjoint regions which has power less than the power of $\mathcal{C}$.
   a. If $A \cap (\bigcup\mathcal{D})$ contains a region then there is a region $D \in \mathcal{D}$ such that $A \cap D$ contains a region.
   b. If $A \cap (\bigcup\mathcal{D})$ contains no region then there is a region $B \subset A$ which is disjoint from every region in $\mathcal{D}$.

The notion of a category base provides the foundation for a general theory of point sets within which more than 100 analogous theorems have been unified. We shall discuss here certain facets of this theory.

We utilize the following generalization of Baire's classification of sets:

DEFINITION. A set S is singular if in every region there is a subregion disjoint from S. A countable union of singular sets is called a meager set. A set which is not meager is called an abundant set. A set is called a Baire set if every region contains a subregion in which either the set or its complement is meager.

For any category base $(X, \mathcal{C})$, the family $\mathfrak{S}(\mathcal{C})$ of singular sets is an ideal, the family $\mathfrak{M}(\mathcal{C})$ of meager sets is a $\sigma$-ideal, and the family $\mathfrak{B}(\mathcal{C})$ is a $\sigma$-field (cf. [12], [20]).

The family $\mathcal{C}_1$ of all closed rectangles in Euclidean space $\mathbf{R}^n$ is a category base for which $\mathfrak{S}(\mathcal{C}_1)$ is the nowhere dense sets, $\mathfrak{M}(\mathcal{C}_1)$ is the first category sets, and $\mathfrak{B}(\mathcal{C}_1)$ is the family of sets which have the Baire property. The family $\mathcal{C}_2$ of all closed sets of positive Lebesgue measure in $\mathbf{R}^n$ is a category base for which both $\mathfrak{S}(\mathcal{C}_2)$ and $\mathfrak{M}(\mathcal{C}_2)$ are the sets of measure zero and $\mathfrak{B}(\mathcal{C}_2)$ is the family of Lebesgue measurable sets. The family $\mathcal{C}_3$ of all perfect sets in $\mathbf{R}^n$ is a category base for which $\mathfrak{S}(\mathcal{C}_3)$, $\mathfrak{M}(\mathcal{C}_3)$, $\mathfrak{B}(\mathcal{C}_3)$ coincide with a classification of sets investigated by Marczewski in [28].

In [27] Marczewski established a general theorem which implies the invariance under the set-theoretical operation $(\mathcal{A})$ of the families $\mathfrak{B}(\mathcal{C}_1)$ and $\mathfrak{B}(\mathcal{C}_2)$. But he resorted to a different method in [28] to establish the fact that the family $\mathfrak{B}(\mathcal{C}_3)$ is closed under operation $(\mathcal{A})$. These three particular results can be obtained as consequences of a single theorem: For any category base $(X, \mathcal{C})$, the family $\mathfrak{B}(\mathcal{C})$ is closed under operation $(\mathcal{A})$.

Suppose now that $h:[0, \infty) \longrightarrow [0, \infty)$ is a monotone increasing, continuous function such that $h(0)=0$ and $h(t) > 0$ if $t > 0$, and let $\mu$ denote the Hausdorff measure associated with h. Assuming the continuum hypothesis, it can be shown that the family $\mathcal{C}_4$ of all closed sets in $\mathbf{R}^n$ of positive Hausdorff measure $\mu$ is a category base. As recently shown, the families $\mathfrak{S}(\mathcal{C}_4)$ and $\mathfrak{M}(\mathcal{C}_4)$ both coincide with the family of sets which have no subset with finite, positive outer measure or, equivalently, each subset of which is a $\mu$-measurable set with measure 0 or $\infty$. The family $\mathfrak{B}(\mathcal{C}_4)$ is the family of $\mu$-measurable sets (cf. [20]).

We note that the property ($\rho$) given above is not generally valid for Hausdorff measures. Assuming the continuum hypothesis, one can show that for each non-$\sigma$-finite Hausdorff measure $\mu$ there exists an uncountable set of infinite measure every subset of which is measurable (cf. [20]). Such sets of infinite measure may be regarded as "negligible," since they are meager sets with respect to the category base $\mathcal{C}_4$. In regard to the property ($\rho$) we note that one can establish in a general way (cf. [15] Theorem 19) that the category bases $\mathcal{C}_1$— $\mathcal{C}_4$ all have the property

($\rho^*$) A set S is a meager set if and only if every subset of S is a Baire set.

Suppose now that n is a natural number and p is a real number with $0 \leq p < n$. Assuming the continuum hypothesis, it can be shown that the family $\mathcal{C}_5$ of all closed sets in $\mathbf{R}^n$ with Hausdorff dimension larger than p is a category base for which both $\mathfrak{S}(\mathcal{C}_5)$ and $\mathfrak{M}(\mathcal{C}_5)$ coincide with the sets which have no subsets of finite, positive Hausdorff q-dimensional outer measure for any number $q > p$; in particular a Borel set is meager if and only if it has Hausdorff dimension $\leq p$. We note that this category base also has the property ($\rho^*$).

Let $\mathcal{E}_5$ denote the family of all sets $E \subset \mathbf{R}^n$ of the form $E = A - Q$, where A is a closed set of Hausdorff dimension > p and Q is an

$\mathfrak{F}_\sigma$-set with Hausdorff dimension $< p$. Assuming the continuum hypothesis, it can be shown that $\mathfrak{E}_5$ is a category base which is equivalent to $\mathfrak{C}_5$; i.e., $\mathfrak{M}(\mathfrak{E}_5) = \mathfrak{M}(\mathfrak{C}_5)$ and $\mathfrak{B}(\mathfrak{E}_5) = \mathfrak{B}(\mathfrak{C}_5)$.

As was pointed out some time ago, there are certain analogies between outer measure and topological dimension (cf. [10], [30]). In this connection, we note that, if $p$ is an integer satisfying $0 < p < n$, then the family $\mathfrak{E}_6$ of all sets $E \subset R^n$ of the form $E = A - Q$, where $A$ is a closed set of topological dimension $> p$ and $Q$ is an $\mathfrak{F}_\sigma$-set with topological dimension $< p$, is completely analogous to $\mathfrak{E}_5$. Assuming the continuum hypothesis, one can show by a similar argument that $\mathfrak{E}_6$ is a category base. In this case, the singular sets (and meager sets) $S$ are characterized by the property that in every region $E$ there is a region $F$ such that the set $S \cap F$ is contained in an $\mathfrak{F}_\sigma$-set of topological dimension $< p$. We note further that the family $\mathfrak{S}(\mathfrak{E}_5)$ has an analogous characterization.

Returning to the Baire category-Lebesgue measure analogies, we first point out that the substitution matrix $\Phi$ can be augmented to form the matrix

$$\Phi^* = \begin{bmatrix} \text{nowhere dense} & , & \text{measure zero} \\ \text{first category} & , & \text{measure zero} \\ \text{second category} & , & \text{positive outer measure} \\ \text{has the Baire property} & , & \text{is measurable} \\ \text{scattered} & , & \text{absolute null} \\ \text{always of first category} & , & \text{absolute null} \\ \text{has the restricted Baire property} & , & \text{is absolutely measurable} \end{bmatrix}$$

We restrict our discussion now to the real line **R**. Let **Z** denote the set of all irrational numbers and let

$$\mathfrak{Z}_1 = \{A \cap Z \ : A \in \mathcal{C}_1\} \ , \quad \mathfrak{Z}_2 = \{A \cap Z \ : A \in \mathcal{C}_2\}.$$

The sets $S \subset R$ which have the restricted Baire property are characterized by

($o_1$) For every order isomorphism $\phi$: $Z \longrightarrow R$ , $S \cap \phi(Z) \in \mathfrak{B}[\phi(\mathfrak{Z}_1)]$.

Analogous to this is the statement

($o_2$) For every order isomorphism $\phi$: $Z \longrightarrow R$ , $S \cap \phi(Z) \in \mathfrak{B}[\phi(\mathfrak{Z}_2)]$,

which characterizes the notion of absolute (or universal) measurability. The restricted Baire property and absolute measurability can also be characterized by the analogous properties

($h_1$) For every homeomorphism $\psi$: $Z \longrightarrow R$ , $S \cap \psi(Z) \in \mathfrak{B}[\psi(\mathfrak{Z}_1)]$,

($h_2$) For every homeomorphism $\psi$: $Z \longrightarrow R$ , $S \cap \psi(Z) \in \mathfrak{B}[\psi(\mathfrak{Z}_2)]$.

Turning to a different type of analogy, we note that there are approximately two dozen theorems concerning order types of linear sets which have valid topological analogues concerning homeomorphism types. For example, the theorem of Dushnik and Miller [3]

(o) There exists a linear set of power of the continuum which is not order isomorphic to a proper subset of itself.

has as analogue the following theorem of Kuratowski [7]:

(h) There exists a linear set of power of the continuum which is not homeomorphic to a proper subset of itself.

Not only are the statements of such theorems analogous, but one can frequently give completely analogous proofs, utilizing Lavrentiev's theorem on extension of homeomorphisms in the topological situation and a corresponding theorem on the extension of order isomorphisms in the ordinal situation (cf. [15] Lemma 5, [16]).

The existence of this ordinal-topological duality justifies us in asserting that the category-measure analogies ($o_1$), ($o_2$) and the category-measure analogies ($h_1$), ($h_2$) are analogous. This state of affairs reminds one of Banach's statement (see [24]) that "a mathematician is one who is able to find analogies between theorems, stronger is one who sees analogies between proofs, better yet is one who perceives analogies between theories, but we can even imagine one who glimpses analogies between the analogies."

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