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Baire Classification of Generalized Extreme Derivatives

Let B_α, respectively L, denote the family of all real Borel functions of a real variable of the class α, respectively the class of all real lebesque measurable functions of a real variable.

W. Sierpiński in [7] showed that the Dini derivatives of a function of B_α are in $B_{\alpha+3}$. S. Banach in [2] proved that the Dini derivatives of a bounded function of B_α are in $B_{\alpha+2}$. L. Misik in [4] showed that the upper, respectively lower Dini derivatives of a function of class B_α are upper, respectively lower semi Borel function of the class $B_{\alpha+1}$. He proved in [5] that the upper unilateral essential derivaties of a continuous function is the limit of a nondecreasing sequence of upper semi Borel functions of the class one and therefore is a lower semi Borel function of the class two. He generalized his result in [6] for $\alpha > 0$.

Theorem (Sierpinski): If F is continuous on $[a,b]$, then each of the Dini derivatives is in Baire class 2.

Proof: We prove the theorem for D^+F, a similar proof holds for D^-F. For each positive integer n let

$$F_n(x) = \sup_{t\in[a,b]} \left\{ \frac{F(t) - F(x)}{t - x} : x + \frac{1}{n+1} \leq t \leq x + \frac{1}{n} \right\}$$

Since F is continuous, each function F_n is also continuous. It is easy to verify that $D^+F(x) = \overline{\lim_{n\to\infty}} F_n(x)$. But an upper limit of a sequence of continuous functions is in Baire Class 2.

A.M. Bruckner, R. O'Malley and B.S. Thomson in [3] introduced the notion of path derivatives. Path derivative is a good setting for exact generalized derivatives, but it is not a good setting for extreme generalized derivatives. B.S. Thomson's simple system is a better setting for extreme generalized derivatives.

Definition: Let $x \in R$, a path leading to x is a set $E_x \subset R$ such that $x \in E_x$ and x is a point of accumulation of E_x. A system of paths is a collection $E = \{E_x : x \in R\}$ such that E_x is a path leading to x.

Definition: Let $F : R \to R$, $E = \{E_x : x \in R\}$ be a system of paths then $F'_E(x) = \underset{\substack{y \to x \\ y \in E_x}}{\underline{\lim}} \frac{F(y) - F(x)}{y - x}$ when the limit exist.

$$\overline{F}'_E(x) = \underset{\substack{y \to x \\ y \in E_x}}{\overline{\lim}} \frac{F(y) - F(x)}{y - x}$$

In the case of extreme path derivatives we would like to imitate Sierpinski's proof, so we face two problems. The first problem is that if we define $F_n(x) = \sup \{ \frac{F(y) - F(x)}{y - x} : y \in E_x \cap [x + \frac{1}{n+1}, x + \frac{1}{n}]\}$ then $E_x \cap [x + \frac{1}{n+1}, x + \frac{1}{n}]$ might be empty, so in that case what should we define for $F_n(x)$? The second problem is that E_x and E_y might behave totally different when x and y are very close, so even for a continuous function it is possible to find a system of paths such that F'_E is not in L.

Definition : Let $E = \{E_x : x \in [0,1]\}$ be a system of paths, each of E_x is compact. If the function $E : x \to E_x$ is a continuous function we say E is a continuous system of paths. (E with Hausdorff's metric forms a metric space).

Lemma: Let $E = \{E_x : x \in [0,1]\}$ be a continuous system of paths then there exist a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n > 0$ for all n. a_n decreasingly tends to zero, and $E_x \cap [x + a_{n+1}, x + a_{n-1}] \neq \emptyset$ for all x.

Theorem 1: Let $F : [0,1] \to R$ be continuous, $E = \{E_x : x \in [0,1]\}$ be a continuous system of paths, then

1) If F is E-differentiable $F'_E \in B_1$.

2) $\overline{F}'_E \in B_2$.

Proof: Let $\{a_n\}_{n=1}^{\infty}$ be the sequence as in lemma, then we define

$$\overline{F}_n(x) = \sup \{\frac{F(y) - F(x)}{y - x} : y \in E_x \cap [x + a_{n+1} , x + a_{n-1}]\}$$

$$\underline{F}_n(x) = \sup \{\frac{F(y) - F(x)}{y - x} : y \in E_x \cap (x + a_{n+1} , x + a_{n-1})\}$$

then $\overline{F}_n(x)$ is an upper semicontinuous function, and $\underline{F}_n(x)$ is a lower semicontinuous function. Since $E_x \cap (x + a_{n+1} , x + a_{n-1}) \cap (x + a_{n+2} , x + a_n) \neq \emptyset$, then the points lost by $\underline{F}_n(x)$ are picked by $\underline{F}_{n+1}(x)$.

Let $g_n(x) = \min(\overline{F}_n(x), \overline{F}_{n+1}(x), \overline{F}_{n-1}(x))$

$h_n(x) = \max (\underline{F}_n(x), \underline{F}_{n+1}(x), \underline{F}_{n-1}(x))$

$g_n(x)$ and $h_n(x)$ are upper semicontinuous and lower semicontinuous respectively, and $g_n(x) \leqq h_n(x)$. So there is a continuous function $P_n(x)$ such that $g_n(x) \leqq P_n(x) \leqq h_n(x)$

1) when F is E-differentiable $F'_E(x) = \underline{\lim}_{n\to\infty} g_n(x) = \underline{\lim}_{n\to\infty} h_n(x) = \underline{\lim}_{n\to\infty} p_n(x)$ so $F'_E \in B_1$.

2) $\overline{F}'_E(x) = \overline{\lim}_{n\to\infty} g_n(x) = \overline{\lim}_{n\to\infty} h_n(x) = \overline{\lim}_{n\to\infty} p_n(x)$ so $\overline{F}'_E \in B_2$.

Similar results hold under certain uniform nonporosity assumptions, but fail to hold without uniform nonporosity.

Example: There is a continuous function $F : [0,1] \to R$ and a nonporous system of paths $E = \{E_x : x \in [0,1]\}$ such that F'_E exist everywhere (possibly infinite) but $F'_E(x)$ is not Borel.

Theorem 2: Let $F : [0,1] \to R$ be in B_α and $E = \{E_x : x \in [0,1]\}$ be a continuous system of paths.

1) If F'_E exist, then F'_E is in $B_{\alpha+2}$.

2) $\overline{F'_E}$ is in L.

For a detailed proof see [1].

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