On Porosity and Exceptional Sets

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In this paper I wish to look again at the role of porosity in the construction of counterexamples. Much has been written about the role of porosity in the equality of derivates of functions. My interest lies in the role of porosity and its several generalizations in the theory of cluster sets. In a early paper I investigated the sets of points on the real line that were the sets on which the cluster sets of various types differed for arbitrary functions -real or complex valued- defined in the half plane $y > 0$.

Using the techniques available to me at the time I was able to show that many of these exceptional sets were of the first Baire category and measure zero. A couple of years later, Yoshida published a paper in the Nagoya Journal [1], extending these results by showing that the sets I had considered were, in fact, sigma-porous sets. I was intrigued by the results, particularly when I discovered that the techniques he used to prove the improved theorems were essentially the same as my earlier techniques. Had I known of the concept of sigma-porosity at the time of my work I might have proved the better result.

The question of whether Yoshida's results were indeed the best possible in this direction is the subject of the work that led to this talk. Some specifics: Let $\nabla_1(x)$ and $\nabla_2(x)$ be two fixed Stolz angles in the upper half plane with vertex at the point x on the real line R; and let the set $E(f, \nabla_1, \nabla_2)$ be the set of all x on R where the function f(x,y) has unequal cluster sets through the two Stolz angles; finally, let E(f) denote the set of points on the real line where f has unequal angular cluster sets on <u>any</u> two Stolz angles.

Yoshida asked the following question: Given an arbitrary sigma-porous set E on R, does there exist a function, f, possibly holomorphic in the upper half plane so that E(f) = E? That is, does sigma-porosity characterize the sets E(f)?

Paul Humke and I attempted to answer this question by constructing just such a function. In that endeavor we were led to yet another variant of the porosity concept that seemed more amenable to the constructive process, that of global porosity. An analysis of the argument used by Yoshida and earlier myself led us to conclude that what was important here was the geometry and regularity of the intervals complimentary to the set E. Since E(f) can be expressed as a countable union of sets of the type $E(f, \nabla_1, \nabla_2)$ we looked closely at the question for fixed Stolz angles. At each point of $E(f, \nabla_1, \nabla_2)$ the angular region ∇_1 is constructed, and the region $G(E,\nabla_1) = \bigcup \nabla_1(x)$, is formed, being the union taken over all x in $E(f,\nabla_1\ \nabla_2)$, One can easily show that at any limit point x_o of E, a sequence can approach x_o through the compliment of $G(E,\nabla_1)$ non-tangentially to the real axis only if the complimentary intervals of E are sufficiently large and regular in a neighborhood of x_o - Yoshida's sigma porosity argument.

What seemed important to us about a set P that was porous, with porosity 1/2, for example, at a point x_o was the fact that x_o should be the limit of intervals of the compliment of E with the property that in any sufficiently small interval I containing x_o the intervals of the compliment of P made up approximately 1/2 of the interval I, and this could be restated in terms of the covering properties of intervals related to these complimentary intervals. In particular, if each of the complimentary intervals were expanded about its midpoint by a factor of 2, such expanded intervals shoud cover most of our interval I, and a like expansion by any factor greater than 2 will cover all of I near x_o. This idea is the one behind the definition of uniform porosity of a set E at a point x. Specifically, let E be a bounded set with a = inf(E) and b = sup(E). Then $(a,b)\setminus cl(E)$ is a union of disjoint open intervals I(n). We let EP(N) denote the set of endpoints of the first N-1 intervals, and r*I(k) be the interval concentric with I(k) whose length is r times the length of I(k). If there is a number r > 0 such that for each N, $E\setminus EP(N)$ is wholly contained in the union of the expanded intervals r*I(k), the union running from k = N to ∞, then E is uniformly porous, and indeed, countable unions of uniformly porous sets actually characterize sigma-porous sets.

We were led to introduce global porosity because of the difficulty encountered in the function-construction phase with the possiblity of needing an infinite collection to do the covering at each point. Accordingly, we modified the definition of uniformly porous in the following way: If for r>0 there is, for each N, an integer N^* such that the set E\ EP(N) is contained in the <u>finite</u> union of intervals r*I(k), the union running from N to N^*, then E is called r-globally porous. A set is just globally porous if it is r-globally porous for some r, and the countable union of globally porous sets is, called sigma-globally porous.

In the paper,"Another Note on Sigma-Porous Sets" [2], we show some characterizations of sigma-globally porous sets. Among other results, we construct there a set which is perfect and porous, but not sigma-globally porous. With this definition in hand we began to attempt to construct a function to partially answer the question of Yoshida. To that end we began with the

LEMMA:

Let E be a $G_{\delta\sigma}$ subset of an r-globally porous set F. Let α be chosen from the interval $(0,\pi/2)$. Then there is a continuous function in the upper half plane to [0,1], and a number β greater that α such that

(i) The cluster set on the Stolz angle $(\alpha, \pi - \alpha)$ = [0,1] for x in E .

(ii) The cluster set on the Stolz angle $(\beta, \pi - \beta)$ = [0] for x in Cl(E).

(iii) The cluster set on any Stolz angle ∇ = [0] for x not in E

From this lemma and the properties of $G_{\delta\sigma}$ sets which are also sigma-globally porous, we were able to make the following first small step in the direction of the question of Yoshida:

THEOREM: If E is a $G_{\delta\sigma}$, sigma-globally porous set on R, then there is a continous function from the upper half plane to [0,1], with the given set E as the exceptional set E(f).

References

[1] Yoshida, H, <u>Tangential Boundadry Properties of Arbitrary Functions in the Unit Disc</u>, Nagoya Mathematics Journal, V.16(1972), pp. 111-120.

[2] Humke,P. and Vessey,T. Another Note on Sigma-Porous Sets, Real Analysis <u>Another Note on Sigma-Porous Sets</u>, Real Analysis Exchange, V. 8#1(1982) pp. 261-271.