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A GENERAL NONSEPARABLE THEORY OF FUNCTIONS AND MULTIFUNCTIONS

Introduction. We present here a new general theory of multifunctions (i.e. set-valued functions) which deals simultaneously with continuity, measurability and the Baire classes of multifunctions. The results on functions will be found to follow directly from the results on multifunctions.

Such a general theory was first developed by Hausdorff, in the case of functions, in his well known monograph "Set Theory" [9]. More recently, in 1972, Kuratowski [15] presented some such results on multifunctions. Both of these works deal, however, only with the case when the range space is separable. Extension of the results to nonseparable range has been an open problem for quite sometime (see e.g. Kuratowski [11]).

Our general theory applies to nonseparable range as well. The first successful attempt in this direction was made by Montgomery [20] in 1935. Using an operation which is known today as "Montgomery operation", he obtained extensions of the results on Baire classes of functions dealing with graph and several variables [13, pp. 378,384]. More recently, in 1971, R. Hansell - a student of A.H. Stone - developed a new interesting tool to tackle this problem. Using his concept of σ-discrete decompositions [7,8], he extended the results on cartesian products of functions in Baire classes and on the continuity of functions in Baire class 1.

The present approach will be found to be considerably simpler, and it applies to practically all the known results in the theory of functions. Besides the extension of known results to nonseparable range, we present several new results on multifunctions like representation, interposition and extension theorems. Two problems of Kuratowski [12] and A.H. Stone [22] regarding the invariance of separability and absolutely Borel sets are also resolved.

The present theory was developed in connection with a new notion of derivaitve [5] which is set-valued; see further [6] for an earlier announcement.

1. Baire systems

Given an arbitrary space X, let $P(X)$ denote the power set of X. We need here two primitive terms to define the necessary structure on X.

A family $L \subset P(X)$ is called a *lattice* of sets in X, or simply a lattice in $P(X)$, if it is closed under the operations $\cup$ and $\cap$ and if $X, \phi \in L$.

Given any lattice L in $P(X)$, it is easy to see that the family L_σ of all countable unions of sets in L is also a lattice in $P(X)$. When $L = L_\sigma$, the lattice L is called a *σ-lattice*.

Given any family of sets $A \subset P(X)$, suppose a family $B \subset P(X)$ is in 1-1 correspondance with A, say $B = \{B_A : A \in A\}$. Then B will be called an *expansion* or a *contraction* of A if $A \subset B_A$ or $A \supset B_A$ respectively for each $A \in A$. Further, if $\{A_i : i \in I\}$ is any partition of the family A, then the family $\{\bigcup_{A \in A_i} A : i \in I\}$ will be called a *coalition* of A.

Next, a class Δ of nonempty families of sets in X will be called a *dispersion* on X if for every family $A \in \Delta$ all the nonempty subfamilies, contractions and coalitions of A are also in Δ.

Given any dispersion Δ on X, it is clear that the class Δ_σ of all countable unions of families in Δ is also a dispersion on X. When $\Delta = \Delta_\sigma$, Δ will be called a *σ-dispersion* on X, and when every family $A \in \Delta$ is disjoint, Δ will be called *disjoint*.

Let us consider some examples of dispersions before coming to the main structure. We define

$$\Delta_o = \{A \subset P(X) : \text{card } A = 1\},$$
$$\Delta_c = \{A \subset P(X) : 0 < \text{card } A \leq \aleph_o\} \text{ and}$$
$$\Delta_1 = \{A \subset P(X) : A \neq \phi\}.$$

It is clear that Δ_o is a dispersion on X, $\Delta_c = \Delta_{o\sigma}$ and that Δ_c and Δ_1

are σ-dispersions on X. Further, Δ_o and Δ_1 are in a sense the minimal and maximal dispersions on X.

Now suppose X is a topological space. Let a family $A \subset P(X)$ be called *fully discrete* or *sparse* if it has a discrete or a locally finite open expansion respectively. We will use Δ_d, $\Delta_{\ell f}$, Δ_{fd} and Δ_s to denote the classes of all nonempty families of sets in X which are discrete, locally finite, fully discrete or sparse respectively. Each of these classes is a dispersion on X, where

$$\Delta_o \subset \Delta_{fd} \subset \Delta_d[\Delta_s] \subset \Delta_{\ell f} \subset \Delta_1 .$$

Further, Δ_o, Δ_{fd} and Δ_d are disjoint.

Next, suppose L is a σ-lattice in $P(X)$ and Δ is any σ-dispersion on X. The pair $S = (L,\Delta)$ will be called a *Baire system* on X provided the following two conditions hold:

(B1) L is Δ-*additive*, viz. if $A \in \Delta$ and $A \subset L$, then $\cup\{A : A \in A\} \in L$,

(B2) Δ is L-*expansive*, viz. every family $A \in \Delta$ has an expansion $E \in \Delta$ such that $E \subset L$.

Further, given a Baire system $S = (L,\Delta)$ on X, if there exists an algebra L_k in $P(X)$ and a dispersion Δ_k on X such that $L = L_{k\sigma}$, $\Delta = \Delta_{k\sigma}$ and L_k is Δ_k-additive, then S will be called a *regular Baire system* on X and $S_k = (L_k,\Delta_k)$ will be called the *kernel* of S.

Clearly, a Baire system $S = (L,\Delta)$ is regular whenever L is a σ-algebra, for then S itself is a kernel of S.

We now present a few important examples of Baire systems on X.

(1) If L is any σ-lattice in $P(X)$, then (L,Δ_c) is a Baire system on X, and if L is an algebra in $P(X)$, then (L_σ,Δ_c) is a regular Baire system with kernel (L,Δ_o).

(2) If X is any topological space with topology T, then (T,Δ_1) and $(T,\Delta_{s\sigma})$ are two Baire systems on X which are usually not regular. However, if X is a 0-dimensional metrizable space, then $(T,\Delta_{s\sigma})$ is regular.

(3) Suppose X is a *perfect space*, viz. a topological space in which every open set is an F_σ-set. Given any countable ordinal α, let $\Sigma_\alpha(X)$ and $\Pi_\alpha(X)$ denote the families of sets in X which are of additive or multiplicative class α respectively. We use further $\mathcal{B}(X)$ and $\mathcal{B}_r(X)$ to denote the families of sets in X which have the Baire or restricted Baire property [13] respectively. The last two families are known to be σ-algebras in $\mathcal{P}(X)$.

The following results are not known under the present hypothesis, but when X is metrizable they can be deduced from a theorem of Montgomery [13, p. 358].

1.1. *Theorem*. Suppose X is a perfect space and $0 < \alpha < \Omega$. Then $(\Sigma_\alpha(X),\Delta_{d\sigma})$, $(\Sigma_\alpha(X),\Delta_{s\sigma})$, $(\mathcal{B}(X),\Delta_{\ell f\sigma})$ and $(\mathcal{B}_r(X),\Delta_{d\sigma})$ are regular Baire systems on X.

2. $\mathcal{L}$-measurability and Δ- and $\mathcal{S}$-continuities

Let $\mathcal{S} = (\mathcal{L},\Delta)$ be any Baire system on X. We call a topology $\mathcal{T}$ on X

(i) a Δ-*topology* if it has an open base $\mathcal{B} \in \Delta$,

(ii) an $\mathcal{S}$-*topology* if it has an open base $\mathcal{B} \in \Delta$ such that $\mathcal{B} \subset \mathcal{L}$.

Thus $\mathcal{T}$ is an $\mathcal{S}$-topology iff it is a Δ-topology which is contained in $\mathcal{L}$.

The following theorem plays a fundamental role in the extension of results to nonseparable spaces.

2.1. *Theorem*. If $\{\mathcal{T}_n\}$ is a sequence of Δ- or $\mathcal{S}$-topologies on X, then so is the topology generated by $\bigcup_n \mathcal{T}_n$.

Next, let $(Y,\mathcal{V})$ be any topological space. Define, for each $V \in \mathcal{V}$,

$$V_* = \{E \subset Y : E \subset V\}, \quad V^* = \{E \subset Y : E \cap V \neq \phi\}.$$

The topology $\tilde{\mathcal{V}}$ generated by $\{V_* : V \in \mathcal{V}\} \cup \{V^* : V \in \mathcal{V}\}$ on $\mathcal{P}(Y)$ is called the *Vietoris topology* on $\mathcal{P}(Y)$.

We will use $\mathcal{M} \equiv \mathcal{M}(X,Y)$ to denote the space of all multifunctions ϕ which map X into $\mathcal{P}(Y)$. Further, $\mathcal{M}^*$, $\mathcal{M}_c$, $\mathcal{M}_k$ and $\mathcal{M}_s$ will denote the sets of multifunctions in $\mathcal{M}$ whose values are nonempty, closed, compact or separable respectively. In case Y is metrizable, M_{cp} is used to denote the set of multifunctions in M whose values are metrically complete, and when Y is a topological vector space, M_{wk} and M_{cv} will denote the sets of multifunctions in M whose values are weakly compact or convex respectively. Further, we set $M^*_k = M^* \cap M_k$, $M_{k,cv} = M_k \cap M_{cv}$, etc.

Now, given a Baire system $S = (L, \Delta)$ on X, a multifunction $\phi \in M$ will be called

(i) L-M (*L-measurable*) if $\phi^{-1}(\tilde{V}) \in L$ for each $\tilde{V} \in \tilde{V}$,

(ii) L-LM (*L-lower measurable*) if $\phi^{-1}(V^*) \in L$ for each $V \in V$,

(iii) L-UM (*L-upper measurable*) if $\phi^{-1}(V_*) \in L$ for each $V \in V$,

(iv) Δ-C (*Δ-continuous*), Δ-LC (*Δ-lower continuous*) or Δ-UC (*Δ-upper continuous*) if it is T-M, T-LM or T-UM respectively for some Δ-topology T on X,

(v) S-C (*S-continuous*), S-LC (*S-lower continuous*) or S-UC (*S-upper continuous*) if it is T-M, T-LM or T-UM respectively for some S-topology T on X.

Thus ϕ is indeed Δ-C, Δ-LC or Δ-UC [S-C, S-LC or S-UC] iff it is continuous, LSC (lower semicontinuous) or USC (upper semicontinuous) relative to some Δ-topology [S-topology] on X (see [13, p. 173] for definitions).

When X is a topological space, ϕ is thus continuous, LSC or USC iff it is $\Sigma_0(X)$-M, $\Sigma_0(X)$-LM or $\Sigma_0(X)$-UM respectively, and ϕ is in $\mathbf{B}_\alpha$ (*Baire class* α), $\mathbf{LB}_\alpha$ (*lower Baire class* α) or $\mathbf{UB}_\alpha$ (*upper Baire class* α) iff it is $\Sigma_\alpha(X)$-M, $\Sigma_\alpha(X)$-LM or $\Sigma_\alpha(X)$-UM respectively. Further, ϕ will be said to have BP (*Baire property*), LBP (*lower Baire property*) or UBP (*upper Baire property*) if it is $B(X)$-M, $B(X)$-LM or $B(X)$-UM respectively. The *restricted Baire properties* RBP, LRBP and URBP of ϕ are defined similarly by replacing $B(X)$ by $B_r(X)$.

Let us now define the L-measurability and Δ- and S-continuities of a function. A function $f : X \to Y$ will be called

(i) L-M if $f^{-1}(V) \in L$ for each $V \in V$,

(ii) Δ-C or S-C if it is T-M for some Δ- or S-topology T respectively on X.

When X is a topological space, f is thus continuous or in $\mathbf{B}_\alpha$ iff it is $\Sigma_0(X)$-M or $\Sigma_\alpha(X)$-M respectively. Further, f has BP or RBP [13] iff it is $\mathcal{B}(X)$-M or $\mathcal{B}_r(X)$-M respectively.

Next, let $\mathbf{R} \equiv (-\infty,\infty)$ and $\overline{\mathbf{R}} \equiv [-\infty,\infty]$ have their usual topologies. A function $f : X \to \overline{\mathbf{R}}$ will be called, further,

(i) $\mathcal{L}$-LM if $\{x \in X : f(x) > c\} \in \mathcal{L}$ for each $c \in \mathbf{R}$,

(ii) Δ-LC or $\mathcal{S}$-LC if it is $\mathcal{T}$-LM for some Δ- or $\mathcal{S}$-topology $\mathcal{T}$ respectively on X,

(iii) $\mathcal{L}$-UM, Δ-UC or $\mathcal{S}$-UC if $-f$ is $\mathcal{L}$-LM, Δ-LC or $\mathcal{S}$-LC respectively.

When X is a topological space, f is thus LSC (in Baire's sense) or in $\mathbf{LB}_\alpha$ iff it is $\Sigma_0(X)$-LM or $\Sigma_\alpha(X)$-LM respectively. Also, f has LBP or LRBP iff it is $\mathcal{B}(X)$-LM or $\mathcal{B}_r(X)$-LM respectively. The upper properties of f are defined analogously.

3. Relations between the properties of functions and multifunctions

In this section we show how the results on functions and closed-valued multifunctions follow from the present theory.

Given any function $f : X \to Y$, let ϕ_f denote the multifunction in $\mathcal{M}$ defined by $\phi_f(x) = \{f(x)\}$, $x \in X$. The results on f, and their extensions to nonseparable Y, follow from the corresponding results on multifunctions due to the following equivalences:

3.1. *Theorem.* If $f : X \to Y$, then

(a) f is $\mathcal{L}$-M $\iff$ ϕ_f is $\mathcal{L}$-LM $\iff$ ϕ_f is $\mathcal{L}$-UM $\iff$ ϕ_f is $\mathcal{L}$-M,

(b) f is Δ-C $\iff$ ϕ_f is Δ-LC $\iff$ ϕ_f is Δ-UC $\iff$ ϕ_f is Δ-C

and

(c) f is $\mathcal{S}$-C $\iff$ ϕ_f is $\mathcal{S}$-LC $\iff$ ϕ_f is $\mathcal{S}$-UC $\iff$ ϕ_f is $\mathcal{S}$-C.

Now, suppose $f : X \to Y$ where $Y = \mathbf{R}$ or $\overline{\mathbf{R}}$. Let ψ_f denote the multifunction in $\mathcal{M}$ defined by $\psi_f(x) = \{y \in Y : y \leq f(x)\}$, $x \in X$. The following theorem indicates how the results on $\mathcal{L}$-LM and $\mathcal{L}$-UM of f follow from the corresponding results on multifunctions:

3.2. *Theorem.* Suppose $f : X \to Y$ where $Y = \mathbf{R}$ or $\overline{\mathbf{R}}$. Then

(a) f is L-LM $\iff \psi_f$ is L-LM $\iff \psi_f$ is S-LC and

(b) f is L-UM $\iff \psi_f$ is L-UM $\iff \psi_f$ is S-UC.

Next, given $\phi \in M$, let $\bar{\phi} \in M$ be defined by $\bar{\phi}(x) = \overline{\phi(x)}$, $x \in X$. In the present theory we do not assume the values of ϕ to be closed in general (see e.g. [3]). The results on closed-valued ϕ, as considered in [14] and [15], follow however from the general case due to the following theorem.

3.3. *Theorem.* (a) $\phi \in M$ is L-LM, Δ-LC or S-LC iff $\bar{\phi}$ is so.

(b) If $\phi \in M_k$ and Y is either regular or a Hausdorff space, then ϕ is L-UM, Δ-UC, S-UC, Δ-C or S-C iff $\bar{\phi}$ is so.

4. Relations between L-measurability and Δ- and S-continuities

We assume from now on that $S = (L,\Delta)$ is a Baire system on X, Y is some topological space and that $\phi \in M(X,Y)$.

If ϕ is L-M, then it is clearly L-LM and L-UM. The converse, however, does not hold in general. But, due to Theorem 2.1,

4.1. *Theorem.* (a) ϕ is Δ-C iff it is Δ-LC and Δ-UC.

(b) ϕ is S-C iff it is S-LC and S-UC.

4.2. *Theorem.* If Y is metrizable and $\phi \in M_k$ is Δ-LC, then ϕ is Δ-C.

Next, if ϕ is S-C, then it is clearly L-M and Δ-C. Similar statements hold for S-LC and S-UC, but the converse of any of these statements does not hold in general. However,

4.3. *Theorem.* Suppose Y is metrizable and that ϕ is Δ-LC.

(a) If $\phi \in M_s$ is L-LM, then it is S-LC.

(b) If $\phi \in M_k$ is L-UM, then it is S-UC.

(c) If $\phi \in M_k$ is L-LM and L-UM, then it is S-C.

Thus Δ-LC turns out to be a regularity condition under which all the results on S-LC, S-UC and S-C multifunctions will hold also for L-LM, L-UM and L-M multifunctions respectively. The next two theorems deal with hypotheses under which this regularity condition holds.

4.4. *Theorem.* If Y is second countable, then ϕ is always Δ_c-LC, and in case $\phi \in M_k$, then it is Δ_c-C.

A metric space X is called *absolutely Borel* or *absolutely analytic* if it is a Borel or analytic set respectively in the completion of X. Thus every absolutely Borel set is absolutely analytic. Let $A(X)$ denote the family of all analytic sets in X. With the help of a theorem of Kaniewski and Pol [10], based on a fundamental lemma of Hansell [7], we obtain

4.5. *Theorem.* Suppose X is absolutely analytic and that Y is metrizable. If $\phi \in M_k$ is $A(X)$-LM, then ϕ is $\Delta_{fd\sigma}$-C.

4.6. *Corollary.* Suppose X is absolutely analytic, $\alpha < \Omega$, $S = (\Sigma_\alpha(X), \Delta_{fd\sigma})$, Y is metrizable and that $\phi \in M_k$.

(a) If $\phi \in \mathbf{LB}_\alpha$, then it is S-LC.

(b) If $\phi \in \mathbf{UB}_\alpha$, then it is S-UC.

(c) If ϕ is in $\mathbf{LB}_\alpha$ and $\mathbf{UB}_\alpha$, then it is S-C.

As it is clear from Theorems 3.1 and 3.2, the above results hold also for functions.

It is important to note here that many of the results will be found to hold only for S-LC, S-UC or S-C multifunctions in general. However, due to Theorems 4.3, 4.4 and 4.5, any such result will hold also for an L-LM, L-UM or L-M multifunction $\phi \in M_k$ respectively provided either (i) Y is second countable, or (ii) X is absolutely analytic, Y is metrizable and $L \subset A(X)$.

5. Properties of L-M and S-C multifunctions

On account of Theorem 2.1, most of the known results on multifunctions in [3], [13], [14] and [15] extend to S-LC, S-UC or S-C (and to Δ-LC, Δ-UC or Δ-C) multifunctions, under suitable hypotheses on X and Y, where S is any Baire system on X and Y need not be separable. A few of these results hold also for L-LM and L-UM multifunctions in general, but others follow from the results on S-LC and S-UC multifunctions in the two particular cases just mentioned.

We present here only two such results as examples, and indicate the nature of other results.

Given any family of multifunctions $\{\phi_i : i \in I\}$ in M, we define the multifunctions $\bigcup_i \phi_i$ and $\bigcap_i \phi_i$ by

$$(\bigcup_i \phi_i)(x) = \bigcup_i \phi_i(x), \quad (\bigcap_i \phi_i)(x) = \bigcap_i \phi_i(x), \quad x \in X.$$

5.1. *Theorem.* If two multifunctions $\phi, \psi \in M$ are both L-LM, L-UM, Δ-LC, Δ-UC, Δ-C, S-LC, S-UC or S-C, then so is $\phi \cup \psi$.

Similar results hold for other operations on multifunctions like $\cap$, $\sim$, cartesian product, vector addition, scalar multiplication, composition, restriction and sequential limit.

Following is an extension of a well-known theorem of Baire [13, p. 394] to multifunctions with arbitrary range.

5.2. *Theorem* (Baire). Let X be a topological space and $S = (\Sigma_1(X), \Delta_{\ell f\sigma})$. If $\phi \in M$ is S-LC, S-UC or S-C, then it is LSC, USC or continuous respectively at a residual set of points in X.

This theorem contains Brisac's result [4] on multifunctions in $\mathbf{LB}_1$ and $\mathbf{UB}_1$ for second countable Y, and also Hansell's generalization [8] of Baire's theorem on functions to nonseparable range.

The result on the graph of a multifunction (see e.g. [15]) holds for an S-UC multifunction whenever Y is metrizable. Further, the Montgomery's version of Lebesgue's result on functions of several variables [13, p. 378] also extends to multifunctions.

6. Some general reduction and selection theorems

We shall assume from now on that $S = (L, \Delta)$ is a regular Baire system on X with kernel $S_k = (L_k, \Delta_k)$. To deal with such a system, some other basic tools are needed besides the unstated results of the last section.

Given a family $A \subset P(X)$, let a contraction R of A be called a *reduction* of A if it is a disjoint family such that $\cup\{A : A \in R\} = \cup\{A : A \in A\}$.

Further, let a base $\mathcal{B}$ of $\mathcal{R}$ be called an *S-base* if $\mathcal{B} \in \Delta$ and $\mathcal{B} \subset \mathcal{L}$. When a reduction $\mathcal{R}$ of $\mathcal{A}$ has some S-base, it will be called an *S-reduction*.

6.1. *Theorem (Reduction theorem).* If a family $\mathcal{A} \subset \mathcal{P}(X)$ has an S-base, then it has an S-reduction $\mathcal{R} \subset \mathcal{L}$.

On choosing S suitably, many of the known reduction theorems follow from this general theorem (see e.g. [13, pp. 279,350], [16, p. 127] and [8]). It also yields a general separation theorem which contains several known results.

6.2. *Theorem (Multiple reduction theorem).* Suppose Δ_k is disjoint. If a point countable family $\mathcal{A} \subset \mathcal{P}(X)$ has an S-base, then it has a sequence of S-reductions $\mathcal{R}_n = \{R_{A,n} : A \in \mathcal{A}\} \subset \mathcal{L}$ $(n = 1,2,\ldots)$ such that $A = \bigcup_n R_{A,n}$ for each $A \in \mathcal{A}$.

6.3. *Theorem (Refinement of partition).* If a partition $\mathcal{P}$ of X has an S-base, then it has a refined partition $\bigcup_n \mathcal{C}_n$ such that each $\mathcal{C}_n$ is a contraction of $\mathcal{P}$, $\mathcal{C}_n \in \Delta_k$ and $\mathcal{C}_n \subset \mathcal{L}_k$.

Following is a general selection theorem which contains similarly the selection theorems of [17], [16, p. 458] and [10]. A function $f : X \to Y$ is called here a *selection* of ϕ if $f(x) \in \phi(x)$ for each $x \in X$.

6.4. *Theorem (Selection theorem).* If Y is metrizable and $\phi \in \mathcal{M}^*_{cp}$ is S-LC, then ϕ admits an S-C selection f.

This theorem leads in turn to generalizations of some of the known results on selections of partitions and of point inverses (see e.g. [16, pp. 463-472]).

The results of the remaining sections are not known for multifunctions in any particular case. Some of these results were announced earlier in [6] for the Baire classes of multifunctions with second countable range.

7. Analytical representation of S-LC and S-UC multifunctions

We present here some representation theorems on multifunctions which are similar in spirit to Baire's representation of real-valued LSC and USC functions in terms of limits of monotone sequences of continuous functions.

Given a sequence of multifunctions $\{\phi_n\}$ in M, let a multifunction $\phi \in M$ be called a (pointwise) *limit* of $\{\phi_n\}$ if, for each $x \in X$ and for each open neighbourhood $\tilde{V}$ of $\phi(x)$ in $P(Y)$ (i.e. $\tilde{V} \in \tilde{V}$), there exists an integer n_x such that $\phi_n(x) \in \tilde{V}$ for $n > n_x$. Here the limit of $\{\phi_n\}$ is not unique in general. However, when Y is T_1, this limit is unique in M_c.

Next, given $\phi, \psi \in M$, we say $\phi \subset \psi$ if $\phi(x) \subset \psi(x)$ for each $x \in X$. The sequence $\{\phi_n\}$ will be called *nondecreasing* or *nonincreasing* if $\phi_n \subset \phi_{n+1}$ or $\phi_n \supset \phi_{n+1}$ respectively for each n.

Let us first state a general result on limits which holds for an arbitrary Baire system S on X.

7.1. *Theorem.* (a) If $\{\phi_n\}$ is nondecreasing, then $\cup_n \phi_n$ is a limit of $\{\phi_n\}$, and it is further L-LM, Δ-LC or S-LC if each ϕ_n is so.

(b) If $\{\phi_n\}$ is nonincreasing and $\phi_1 \in M_k$, then $\cap_n \phi_n$ is a limit of $\{\phi_n\}$, and it is further L-UM, Δ-UC or S-UC if each ϕ_n is so.

When a multifunction $\phi \in M$ is of the form $\bigcup_{i=1}^{n} \phi_{f_i}$ where $\{f_i : i = 1,2,\dots,n\}$ is any finite set of functions from X to Y, ϕ will be called an *elementary multifunction* and f_i's will be called the *elements* of ϕ. It should be noted here that an elementary multifunction ϕ is S-C whenever each of its elements is so (see Theorem 5.1).

7.2. *Theorem.* Suppose Δ_k is disjoint, Y is metrizable and $\phi \in M^*_{s,cp}$. Then ϕ is S-LC iff it is a limit of a nondecreasing sequence of elementary multifunctions $\{\phi_n\}$ whose elements are S-continuous.

The above theorem does not hold for the ordinary continuity in general. But when ϕ is convex-valued, we obtain the following result with the help of two selection theorems of Michael [19] which are in turn contained in this result.

7.3. *Theorem.* Let Y be a metrizable locally convex space and $\phi \in M^*_{cv}$. Suppose either

(i) X is collectionwise normal and $\phi \in M_k$, or

(ii) X is perfectly normal, Y is separable and $\phi \in M_{cp}$.

Then ϕ is LSC iff it is a limit of a nondecreasing sequence of elementary multifunctions $\{\phi_n\}$ whose elements are continuous.

The Baire's representations of real-valued LSC and USC functions follow clearly from this theorem with the help of Theorem 3.2. Theorem 7.2 yields in turn similar representations of real-valued S-LC and S-UC functions, and on choosing $S = (\Sigma_\alpha(X), \Delta_c)$ these results yield Hausdorff's representations [9] of real-valued functions in $\mathbf{LB}_\alpha$ and $\mathbf{UB}_\alpha$ $(\alpha > 0)$.

The next two theorems deal with the representation of S-UC and USC multifunctions. These theorems are quite different from the above results since there is no duality between S-LC and S-UC multifunctions. Let $\phi \in M$ be called S-TC (*S-totally continuous*) if it is S-C relative to the discrete topology on $P(Y)$.

7.4. *Theorem.* Suppose Δ_k is disjoint, Y is metrizable and that $\phi \in M_k$ is Δ-C. Then ϕ is S-UC iff it is a limit of a nonincreasing sequence of S-TC multifunctions $\{\phi_n\}$.

Moreover, if Y is locally compact, then $\phi_n \in M_k$ for each n.

When Y is a topological vector space, we call $\phi \in M$ *weakly* USC if it is USC relative to the weak topology of Y, and ϕ is called *weakly continuous*, or a *weak limit* of a sequence $\{\phi_n\}$ in M, if it is continuous, or a limit of $\{\phi_n\}$, relative to the Vietoris topology on $P(Y)$ which is determined by the weak topology of Y. Further, Y* denotes the continuous dual of Y.

7.5. *Theorem.* Let X be perfectly normal and Y a locally convex Hausdorff space. Suppose $\phi \in M^*_{cv}$ is weakly USC and that either

(i) Y is barrelled, Y^* is weak* separable and $\phi \in M_k$, or

(ii) Y^* is strongly separable.

If there exists a weakly continuous multifunction $\psi \in M_{wk}$ such that $\phi \subset \psi$ (or, in particular, X is locally compact and paracompact and $\phi \in M_{wk}$), then ϕ is a weak limit of a nonincreasing sequence of weakly continuous multifunctions $\{\phi_n\}$ in M_{wk}.

8. Analytical representation of multifunctions in $\mathbf{B}_\alpha$

We present here two representation theorems on multifunctions in $\mathbf{B}_\alpha$, with $\alpha > 1$ and $\alpha = 1$, in terms of limits of elementary multifunctions in lower Baire classes. These results lead in turn to generalized versions of the classical Baire-Lebesgue-Hausdorff theorem for multifunctions and functions with arbitrary range.

8.1. *Theorem.* Suppose X is a perfect space, Y is metrizable, $\phi \in M_k^*$ is $\Delta_{s\sigma}$-LC and that $\alpha > 1$. If $\phi \in \mathbf{B}_\alpha$, then it is a limit of a sequence of elementary multifunctions $\{\phi_n\}$ whose elements are in Baire classes lower than α.

Moreover, if $\alpha = \lambda + 1$ where λ is a limit ordinal, then the elements of each ϕ_n are in Baire classes lower than λ.

8.2. *Theorem.* Suppose X is perfectly normal, Y is a metrizable absolute retract and that $\phi \in M_k^*$ is $\Delta_{s\sigma}$-LC. If $\phi \in \mathbf{B}_1$, then it is a limit of a sequence of elementary multifunctions $\{\phi_n\}$ whose elements are continuous.

The above theorems yield similar results on functions which generalize some of the known results (see e.g. [13, pp. 390, 391], [2] and [8]).

Now, let the analytic classes of multifunctions be defined by transfinite induction as in the case of functions [13, p. 392].

8.3. *Theorem* (Baire-Lebesgue-Hausdorff). Suppose X is perfectly normal, Y is a metrizable absolute retract and that $\phi \in M_k^*$ is $\Delta_{s\sigma}$-LC. Then ϕ is in analytic class α iff it is in $\mathbf{B}_\alpha$ or $\mathbf{B}_{\alpha+1}$ according as α is finite or infinite.

The same holds for a $\Delta_{s\sigma}$-C function $f : X \to Y$ in terms of its analytic class as a function.

Since every convex subset of a metrizable locally convex space is an absolute retract, the above theorem generalizes its various known versions for functions to nonseparable range (see Baire [1], Lebesgue [18], Hausdorff [9], Kuratowski [13, p. 393] and Banach [2]; Hansell [8] gave another version of this theorem for nonseparable range but his proof is unfortunately incomplete).

9. Interposition theorems

We present here two interposition theorems on multifunctions which are similar in spirit to the interposition theorem of Hahn [9, p. 281] on real-valued functions.

9.1. *Theorem.* Let Δ_k be disjoint and Y be metrizable. Suppose $\phi \in M_k$ is Δ-C and S-UC, $\psi \in M$ is S-LC and that $\phi \subset \psi$.

(a) If $\psi \in M_{s,cp}$, then there exists an S-C multifunction $\theta \in M$ such that $\phi \subset \theta \subset \psi$.

(b) If Y is locally compact, then there exists an S-C multifunction $\theta \in M_k$ such that $\phi \subset \theta \subset \psi$.

Moreover, if $\psi \in M^*_{cp}$, then in each case $\theta \in M^*$.

The above theorem does not hold for the ordinary continuity in general. In the case of convex-valued multifunctions we have, however,

9.2. *Theorem.* Let X be perfectly normal and Y a separable reflexive normed space. Suppose $\phi \in M_{cv}$ is USC, $\psi \in M_{cv}$ is LSC and $\phi \subset \psi$. If either (i) $\phi \in M_k$ and $\psi \in M^*$, or (ii) $\phi \in M^*_{wk}$ and Y* is separable, then there exists a weakly continuous multifunction $\theta \in M^*_{wk,cv}$ such that $\phi \subset \theta \subset \psi$.

Hahn's interposition theorem follows easily from this theorem with the help of Theorem 3.2. Theorem 9.1 yields in turn a similar interposition theorem on real-valued functions, and on choosing $S = (\Sigma_\alpha(X), \Delta_c)$, $\alpha > 0$, this result yields the interposition theorem of Hausdorff [9, p. 294] on Baire classes α of real-valued functions.

10. Extension theorems

We present here some extension theorems on multifunctions which yield generalizations of known extension theorems on functions to nonseparable range.

Given any Baire system $S = (L,\Delta)$ on X and a set $E \subset X$, set $L_E = \{A \cap E : A \in L\}$ and $\Delta_E = \{A \in \Delta : A \subset P(E)\}$. Then $S_E = (L_E,\Delta_E)$ is clearly a Baire system on E. Further, if S has a kernel $S_k = (L_k,\Delta_k)$, then $S_{kE} = (L_{kE},\Delta_{kE})$ is a kernel of S_E. Hence if S is regular, so is S_E.

The dispersions Δ_d, Δ_s, Δ_{fd} and $\Delta_{\ell f}$ are defined, however, in terms of a given topology on X. Hence in these cases it is more natural to define Δ_E directly in terms of the relative topology of E. The following results hold also for this definition of Δ_E provided X is metrizable.

If $\phi \in M$, let ϕ_E denote the restriction of ϕ to E. The first two extension theorems are obtained with the help of the interposition theorems of §9.

10.1. *Theorem.* Let Δ_k be disjoint, Y be metrizable and $E \subset X$ such that $X \sim E \in L$. Suppose $\phi \in M_k(E,Y)$ is S_E-C and that either (i) X is metrizable or (ii) Y is separable.

(a) If there exists some S-LC multifunction $\psi \in M_k(X,Y)$ such that $\phi \subset \psi_E$, then ϕ has an S-C extension $\phi^* \in M_k$ to X, and in case $\phi, \psi \in M^*$, then $\phi^* \in M^*$.

(b) If Y is locally compact, then ϕ has an S-C extension $\phi^* \in M_k$ to X. Further, if there exists some Δ-C and S-UC multifunction $\theta \in M_k^*(X,Y)$ such that $\theta_E \subset \phi$, then $\phi^* \in M^*$.

10.2. *Theorem.* Let F be a closed subset of a perfectly normal space X, Y a separable reflexive normed space and $\phi \in M_{cv}^*(F,Y)$ be continuous. Suppose either (i) $\phi \in M_k$, or (ii) $\phi \in M_{wk}$, Y* is separable and there exists some USC multifunction $\psi \in M_{wk,cv}^*(X,Y)$ such that $\psi_F \subset \phi$. Then ϕ has a weakly continuous extension $\phi^* \in M_{wk,cv}^*$ to X.

Using Theorem 3.1 and S - $(\Sigma_\alpha(X),\Delta_{d\sigma})$, $\alpha > 0$, Theorem 10.1 yields a generalization of Hausdorff-Kuratowski extension theorem [13, p. 434] on functions in $\mathbf{B}_\alpha$ to nonseparable range. Theorem 10.2 is in turn a multi-functional version of Tietz extension theorem, but the two results are not comparable.

The following theorem generalizes similarly another extension theorem of Kuratowski [13, p. 434] on functions to nonseparable range. (Hansell [8] stated a partial result in this direction but his proof contains some unfortunate errors).

10.3. *Theorem.* Suppose X is metrizable, Y a complete metric space, $E \subset X$ and that $\alpha > 0$. If a $\Delta_{s\sigma}$-C multifunction $\phi \in M_k(E,Y)$ is in $\mathbf{B}_\alpha$, then it has a $\Delta_{s\sigma}$-C extension $\phi^* \in M_k$ in $\mathbf{B}_\alpha$ to a set in $\Pi_{\alpha+1}(X)$, and in case $\phi \in M^*$, then $\phi^* \in M^*$.

Moreover, if X is any first countable space, the same holds for $\alpha = 0$.

Following is a generalization of Lavrentiev's extension theorem [13, p. 436] to nonseparable spaces. A bijection f between X and Y is called a *homeomorphism of class* (α,β) if $f \in \mathbf{B}_\alpha$ and $f^{-1} \in \mathbf{B}_\beta$. Let f be called, further, Δ-*bicontinuous* if f and f^{-1} are both Δ-C.

10.4. *Theorem* (Lavrentiev). Let X and Y be two complete metric spaces. If f is a $\Delta_{s\sigma}$-bicontinuous homeomorphism of class (α,β) from a set $A \subset X$ onto a set $B \subset Y$, then it has an extension to a $\Delta_{s\sigma}$-bicontinuous homeomorphism of the same class between two sets $A' \in \Pi_{\alpha+\beta+1}(X)$ and $B' \in \Pi_{\beta+\alpha+1}(Y)$ such that $A \subset A'$ and $B \subset B'$.

11. The invariance of separability and of absolutely Borel sets

Finally, we present solutions of two problems of Kuratowski and A.H. Stone on the invariance of above mentioned properties.

Kuratowski raised the question in [12, p. 399] whether separability is invariant under Borel measurable mappings between metric spaces. The answer is known to be affirmative under the continuum hypothesis [13, p. 399].

11.1. *Theorem.* Suppose f maps a separable space X onto a metrizable space Y. Then Y is separable iff f is Δ_c-C, or, equivalently, iff f is $\Delta_{\ell f\sigma}$-C.

Consequently, if f is Borel measurable and X is absolutely analytic, then Y is separable.

A bijection f between X and Y is called a *Borel isomorphism* if f and f^{-1} are both Borel measurable. A.H. Stone raised the question in [22] whether absolutely Borel sets are invariant under Borel isomorphisms between metric spaces. The following solution is obtained from Theorem 10.3 with the help of a result of Preiss [21].

11.2. *Theorem.* Suppose f is a Borel isomorphism between two metrizable spaces X and Y where X is absolutely Borel. Then Y is absolutely Borel iff f^{-1} is $\Delta_{\ell f\sigma}$-C.

Consequently, if Y is absolutely analytic, then it is absolutely Borel.

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