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A point about $\mathbb{N}\times\mathbb{N}$ matrices and ℓ^∞.

by

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When we assembled one afternoon during the Waterloo Symposium, the following question from Marshall Ash was on the blackboard ($m \wedge n$ means $\min(m,n)$):

If $\sum_{j=1}^{\infty} a_j = 0$ and $\left|\sum_{k=1}^{n} b_k\right| \leq M < \infty$, does

$$\lim_{m\wedge n\to\infty} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{j}{\sqrt{j^2+k^2}} a_j b_k \quad \text{exist?}$$

The answer is: not always. My explanation may be the longest, since it involves putting the question into a setting involving the sequence spaces c_0 and ℓ^∞. In this more familiar setting an application of the fact that, in ℓ^1, weak and strong convergence of sequences are equivalent (Dunford-Schwartz [1, p. 296]) reduces the problem to checking whether an operator T (defined later) maps c_0 into ℓ^1 continuously. Finally, an example shows that it does not, and this gives the negative answer.

Let us denote by A_m the partial sum $\sum_{j=1}^{m} a_j$, $m = 1,2,\ldots$, so $A_m \to 0$ as $m \to \infty$, or, $A \in c_0$. Similarly, $B \in \ell^\infty$, where B_n denotes the n-th partial sum of the b_k. It will help to use $F(x,y) = x/\sqrt{x^2+y^2}$. Then, with $S_{mn}(a,b)$ denoting the double sum in the question, we do enough summing by parts (Zygmund [2,p.3]) to get, for large m and n,

$$S_{mn}(a,b) = \sum_{k=1}^{n-1} \sum_{j=1}^{m-1} [F(j,k) - F(j,k+1) - F(j+1,k) + F(j+1,k+1)]A_j B_k$$

$$(1) \qquad + B_n \sum_{j=1}^{m-1} [F(j,n) - F(j+1,n)]A_j + \text{(a similar term)}$$

$$+ A_m F(m,n) B_n \ .$$

The "similar term" arises by interchanging A and B, m and n, j and k. We will show that the first (double) sum "replaces" S_{mn}. The last term clearly tends to 0 as $m \wedge n \to \infty$. The first of the "similar terms" may be written as

$$(2) \qquad -B(n) \int_1^{m-1} \frac{\partial F}{\partial x}(x,n) A(x)\,dx \ ,$$

where $A(x) = A_j$ in $[j, j+1)$. Noting that $F(x,y) = x/r$, we have $\partial F / \partial x = y^2/r^3$, so the quantity in (2) is dominated by

$$|B(n)| \int_0^\infty \frac{n^2}{(x^2+n^2)^{3/2}} |A(x)|\,dx = |B(n)| \int_0^\infty \frac{1}{(x^2+1)^{3/2}} |A(nx)|\,dx \ ,$$

which is bounded (independent of m), and, by Lebesgue's dominated convergence theorem, tends to 0 as $n \to \infty$. The other term is a little easier, since the corresponding integral only has to be shown to be (uniformly) bounded.

Therefore, we have shown that $S_{mn}(a,b)$ has a limit as $m \wedge n \to \infty$, if and only if the same is true for $L_{mn}(A,B) = \sum_{j=1}^{m} \sum_{k=1}^{n} G_{kj} A_j B_k$, in which G_{kj} denotes the quantity in square brackets in the double sum in (1).

We might now apply the following lemma, but will wait until after its proof to do so.

Lemma:

Let $\mathfrak{X}$ be a topological linear space of sequences $A = \{A_m\}$, of second category in itself. Suppose that $\chi_m A \to A$ in $\mathfrak{X}$ as $m \to \infty$, where $(\chi_m A)_j = A_j$ for $1 \le j \le m$, and $(\chi_m A)_j = 0$ otherwise. Then for any matrix $(K_{kj})_{k \ge 1, j \ge 1}$,
$\lim_{m \wedge n \to \infty} \sum_{k=1}^{n} \sum_{j=1}^{m} K_{kj} A_j B_k$ exists for each $A \in \mathfrak{X}$ and $B \in \ell^\infty$, if and only if the linear operator T given by $(TA)_k = \sum_{j=1}^{\infty} K_{kj} A_j$, $k = 1,2,\ldots$, maps $\mathfrak{X}$ continuously into ℓ^1, in which case the limit is given by $\sum_{k=1}^{\infty} \{ \sum_{j=1}^{\infty} K_{kj} A_j \} B_k$.

Proof: Let $L_{mn}(A,B)$ denote the double sum, and let $\langle x, x^* \rangle$ denote the duality pairing between a space Y and its dual space Y^*. Then, if $L(A,B) = \lim_{m \wedge n \to \infty} L_{mn}(A,B)$ exists for each $A \in \mathfrak{X}$ and $B \in \ell^\infty$, we have, for each fixed A, that $L(A,B) = \langle B, X \rangle$ for some $X = X(A) \in (\ell^\infty)^*$. It is straightforward to show that, if $\langle \chi_n B, X \rangle$ converges to $\langle B, X \rangle$, then $\langle B, X \rangle = \sum_{k=1}^{\infty} B_k X_k$, where $\sum_{k=1}^{\infty} |X_k| < \infty$.

Now $L(\chi_m A, \chi_n B) = L_{mn}(A,B)$ and $L(A, \chi_n B) = \lim_{m \to \infty} L(\chi_m A, \chi_n B)$. We examine

$$|L(A,B) - L(A,\chi_n B)| \le |L(A,B) - L(\chi_m A, \chi_n B)| + |L(\chi_m A, \chi_n B) - L(A, \chi_n B)| .$$

The first term on the right is small if both of m,n are large enough. Having chosen m and n, we may further restrict m, to ensure that the second term is small. It follows that $L(A,\chi_n B) = \langle \chi_n B, X \rangle \to \langle B, X \rangle = L(A,B)$, so $L(A,B) = \langle TA, B \rangle$, where T is a linear operator mapping $\mathfrak{X}$ into ℓ^1 (here, the pairing is $\langle \ell^1, \ell^\infty \rangle$).* Since $L_{mn}(A,B) = \langle \chi_n T \chi_m A, B \rangle$, $\chi_{n_r} T \chi_{m_r} A$ converges weakly to TA if $m_r \wedge n_r \to \infty$ as $r \to \infty$. Since sequential weak and strong convergence in ℓ^1 are equivalent, the convergence is actually in ℓ^1 norm. Since each $\chi_n T \chi_m$ is bounded from $\mathfrak{X}$ to ℓ^1, the principle of uniform boundedness shows that T is a continuous operator, as was to be shown. Since $\chi_m A \to A$ in $\mathfrak{X}$, $(TA)_k = \lim_{m\to\infty} (T\chi_m A)_k =$

$$= \lim_{m\to\infty} \langle T\chi_m A, e_k \rangle = \lim_{m\to\infty} \sum_{j=1}^{m} K_{kj} A_j ,$$

so the series converges, and the lemma follows.

Now if we assume that the limit $L(A,B)$ exists, we must agree that $\{\sum_{j=1}^{\infty} G_{kj} A_j\}$ is a sequence in ℓ^1 whenever $A \in c_0$, where $G_{kj} = F(j,k) - F(j,k+1) - F(j+1,k) + F(j+1,k+1)$. Let us take $A_m = \frac{1}{n+1}$ if $2^n \le m < 2^{n+1}$, $n \ge 0$. Then

$$(TA)_k = \sum_{n=0}^{\infty} \frac{1}{n+1}[F(2^{n+1},k+1) - F(2^{n+1},k) - F(2^n,k+1) + F(2^n,k)],$$

because of telescoping on the dyadic blocks. Let us sum by parts- since F is bounded, the boundary term will tend to zero- giving

$$(TA)_k = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}[F(2^{n+1},k+1) - F(1,k+1) - F(2^n,k) + F(1,k)].$$

Since $\{F(1,k) - F(1,k+1)\} \approx \{1/k^2\} \in \ell^1$, it is enough to show that when these terms are dropped the remaining part is not in ℓ^1:

* this easy argument could have been avoided by citing weak sequential completeness of ℓ^1.

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\left(\frac{2^{n+1}}{\sqrt{2^{2n+2}+(k+1)^2}} - \frac{2^n}{\sqrt{2^{2n}+k^2}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{(n+1)(n+2)} \frac{2A^{1/2}-B^{1/2}}{A^{1/2}\ B^{1/2}} \quad (A = 2^{2n}+k^2, B = 2^{2n+2}+(k+1)^2)$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{(n+1)(n+2)} \frac{4A-B}{(2A^{1/2}+B^{1/2})A^{1/2}B^{1/2}}$$

$$= \sum_{n=0}^{\infty} \frac{2n}{(n+1)(n+2)} \frac{3k^2-2k-1}{(2A^{1/2}+B^{1/2})\ A^{1/2}B^{1/2}} \geq 0$$

for each $k \geq 1$.

Since, for n fixed, $A^{1/2}$ and $B^{1/2}$ are asymptotic to k as $k \to \infty$, we have, that the sum in k is like $\Sigma\, 1/k$, so $TA \notin \ell^1$.

References

1. N. Dunford, and J. Schwartz, Linear Operators, Part I, Interscience, New York, 1958.

2. A.Zygmund, Trigonometric Series, vol. I, Cambridge, 1968.

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