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Hájek's Theorem Does Not Hold for $n > 1$

1. In [3], O. Hájek proved that the extreme bilateral derivatives of an arbitrary real function of a real variable are in the second class of Baire. An analogous theorem for extreme strong derivatives of an additive interval function defined on E_n does not hold for $n > 1$.

2. Let E_n be n-dimensional Euclidean space, $d(A)$ the diameter of A, and $m(A)$ the Lebesgue outer measure of the subset A of E_n. Let $(\mathcal{X}_n, \rho)$ be the metric space of all non-degenerate closed intervals in E_n, where the metric $\rho(I,J)$, $I, J \in \mathcal{X}_n$ is defined by the symmetric difference $I \Delta J$ of I and J as follows: $\rho(I,J) = m(I \Delta J)$.

Let φ be an additive interval function defined on $(\mathcal{X}_n, \rho)$ (or on some suitable subset of $\mathcal{X}_n$). Then the upper strong derivative $\bar{\varphi}'(X)$ of φ at X is defined as follows:

$$\bar{\varphi}'(X) = \inf\{\sup\{\frac{\varphi(I)}{m(I)}: X \in I,\ I \in \mathcal{X}_n,\ d(I) \leq \frac{1}{k}\} : k =$$

$$= 1, 2, 3, \ldots\}.$$

Proposition. There exists an additive interval function defined on $(\mathcal{X}_2, \rho)$ whose upper strong derivative is not Borel measurable.

Proof. Let $C = \{(x,y) \in E_2 : x > 0,\ y > 0,\ x^2 + y^2 = 1\}$. Let f be the characteristic function of a subset A of C

which is not Borel measurable. Let $\varphi : \mathcal{X}_2 \to (-\infty, \infty)$ be defined as follows: $\varphi([a,b] \times [c,d]) = f(b,d) - f(a,d) - f(b,c) + f(a,c)$ for each $[a,b] \times [c,d] \in \mathcal{X}_2$, where $a < b$ and $c < d$. The function φ is an additive interval function defined on $\mathcal{X}_2$.

Let $I = [a,b] \times [c,d]$ be sufficiently small. Let $X \in A$. If $X = (a,c)$ or $X = (b,d)$, then $\varphi(I) = 1$; if $X = (a,d)$ or $X = (b,c)$, then $\varphi(I) = -1$ or $\varphi(I) = -2$. Therefore $\bar{\varphi}'(X) = \infty$. Let $X \notin A$. If $X = (a,c)$ or $X = (b,d)$, then $\varphi(I) = 0$; if $X = (a,d)$ or $X = (b,c)$, then $\varphi(I) = 0$ or $\varphi(I) = -1$. Therefore $\bar{\varphi}'(X) = 0$. Therefore $\bar{\varphi}'$ is not Borel measurable.

3. In the paper [5], it is proved that the upper strong derivative of each continuous additive interval function defined on $(\mathcal{X}_n, \rho)$ is of the second class of Baire.

Let T be the set of all $(i_1, \dots, i_n)$, where $i_j \in \{-1, 1\}$ for all $j = 1, 2, \dots, n$. Let $E_n^+ = \{(h_1, \dots, h_n) \in E_n : h_j > 0$ for $j = 1, 2, \dots, n\}$. Let $i = (i_1, \dots, i_n) \in T$, $X, Y \in E_n$ and $A \subset E_n$. Then we define $X + iY = (x_1 + i_1 y_1, \dots, x_n + i_n y_n)$ and $X + iA = \{X + iZ : Z \in A\}$. We can define extreme "unilateral" strong derivatives. Let φ be an interval function defined on $(\mathcal{X}_n, \rho)$ and let $i \in T$. Then the upper i-strong derivative $\bar{\varphi}^{(i)}(X)$ of φ at X is defined as follows: $\bar{\varphi}^{(i)}(X) = \inf\{ \sup\{\frac{\varphi(I)}{m(I)} : I = \langle \min(X,Y), \max(X,Y) \rangle, Y \in X + i E_n^+, d(I) \leq \frac{1}{k}\} : k = 1, 2, 3, \dots\}$. By $\min(X,Y)$ or $\max(X,Y)$ we understand the point

$(\min(x_1,y_1),\ldots,\min(x_n,y_n))$ or $(\max(x_1,y_1),\ldots,\max(x_n,y_n))$, respectively and $\min(X,Y)$ and $\max(X,Y)$ are the principal vertices of the interval I.

A function $f : E_n \to (-\infty,\infty)$ will be called lower T-semicontinuous iff for each $X \in E_n$ and for each $a \in (-\infty,\infty)$ satisfying the condition $f(X) > a$ there exists an $i \in T$ and $Y \in X + iE_n^+$ such that $f(Z) > a$ for all Z of the closed interval $I = \langle \min(X,Y), \max(X,Y) \rangle$.

In [5], it is also proved that (i) the upper i-strong derivative of a continuous interval function defined on $(\mathcal{X}_n, \rho)$ is the limit of a non-increasing sequence of lower semicontinuous functions and hence it is in the second class of Baire; (ii) the upper strong derivative of a subadditive interval function is the limit of a non-increasing sequence of lower T-semicontinuous functions. Each lower T-semicontinuous function is Lebesgue measurable. This is a consequence of the known assertion that the union of an arbitrary system of closed intervals is a Lebesgue measurable set (Lemma 4.1 of [2], p. 112, or [6], p. 177.)

From our Proposition and (ii) we have that there are lower T-semicontinuous functions which are not Borel measurable.

4. In Banach's proof, [1] (as in my proof, [4]), that extreme unilateral derivatives of each bounded (arbitrary) Borel function of a real variable of the class α are Borel functions of the class $\alpha + 2$, the following assertion plays a key role: Let f be a real Borel function of a real variable of the class α, where $\alpha > 0$, let $0 < a < b$ (let

$0 < a < b$ and k a natural number). Then the function $\varphi(x;a,b) = \sup\{f(x+h) - f(x) : a \leqq h \leqq b\}$ ($\varphi_k(x;a,b) = \sup\{f(x+h) - f(x) : |f(x+h)\cdot| \leqq k,\ a \leqq h \leqq b\}$) is a Borel function of the class α ([1], ([4])).

To prove the last mentioned assertion, S. Banach proved first that the function $f(x) + \varphi(x;a,b)$ has left and right limits at every real number. From the asymmetry theorem of W.H. Young it follows that $f(x) + \varphi(x;a,b)$ is of the first class of Baire. In E_1 the asymmetry is related to countability, but in E_n for $n > 1$ it is related to sets of the first category and of Lebesgue measure zero.

Open questions:

1. Does there exist a Borel additive interval function φ defined on $(\mathcal{X}_n, \rho)$ of the first class for which the upper strong derivative $\bar{\varphi}'$ is not a Borel function?

2. Let $n > 1$, let φ be a Borel additive interval function defined on $(\mathcal{X}_n, \rho)$ of the class α, $\alpha > 0$, $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n) \in E_n^+$ such that $a_i < b_i$ for $i = 1, 2, \dots, n$. Let $\varphi_k(X;A,B) = \sup\{\varphi(\langle X, X+H\rangle) : |\varphi(\langle X, X+H\rangle)| \leqq k,\ H = (h_1, \dots, h_n),\ a_i \leqq h_i \leqq b_i$ for $i = 1, 2, \dots, n\}$. Is $\varphi_k(X;A,B)$ a Borel function of the class α?

3. Is it true that $\bar{\varphi}^{(i)}$ is a Borel function of the class $\alpha + 2$ if φ is a Borel additive interval function of the class α? We assume $n > 1$. For $n = 1$, this

assertion is true.

References

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