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ON THE FIRST AND THE FIFTH CLASS OF ZAHORSKI

Introduction. For a real-valued function of a real variable f, the associated sets of f are the sets $E^r(f) = \{x: f(x) < r\}$ and $E_r(f) = \{x: f(x) > r\}$ where r is real. It is well-known that f is in the first Baire class ($\mathcal{B}_1$) if and only if every associated set of f is of type F_σ. In [8], Zahorski considered a hierarchy $\{\mathfrak{M}_i\}_{i=0}^{5}$ of subclasses of $\mathcal{B}_1$ $(\mathfrak{M}_{i-1} \supset \mathfrak{M}_i)$. Each of these classes is defined in terms of associated sets: f is in $\mathfrak{M}_i$ if and only if every associated set of f is in M_i where M_i is a certain family of F_σ sets. Zahorski showed that $\mathfrak{M}_0 = \mathfrak{M}_1 = \mathcal{D}\mathcal{B}_1$ (the class of all Darboux-Baire 1 functions) and $\mathfrak{M}_5 = \mathcal{A}$ (the class of all approximately continuous functions).

Let $\mathcal{H}$ denote the class of all homeomorphisms of the real line R onto itself. A theorem of Maximoff [5] asserts that for any function $f \in \mathfrak{M}_1$ there exists $h \in \mathcal{H}$ such that $f \circ h \in \mathfrak{M}_5$. Gorman [2] showed that a set analogue of this theorem holds: If $E \in M_1$, then there exists $h \in \mathcal{H}$ such that $h(E) \in M_5$.

In Theorem 1 of this paper we characterize all countable collections $S \subset M_1$ for which there exists $h \in \mathcal{H}$ such that $\{h(E): E \in S\} \subset M_5$. The idea is based on a lemma due to Preiss [7] ([7] contains a proof of Maximoff's theorem). Maximoff's theorem is then stated as a simple corollary of Theorem 1.

It is known that $\mathcal{A} = \mathfrak{M}_5$ is exactly the class of continuous functions relative to a certain topology (the density topology) in the domain space. Thus a number of results concerning $\mathfrak{M}_5$-functions can be obtained by topological methods. No such topology exists for $\mathcal{DB}_1 = \mathfrak{M}_1$. Applying Theorem 1, we show that some of these results (two lemmas of Zahorski [8], extension theorems [6], [4]) have valid analogues in $\mathfrak{M}_1$.

<u>Notations</u>. In what follows, all sets dealt with are subsets of R and all functions, unless otherwise specified, have R as domain. N denotes the set of all natural numbers, λ the Lebesgue measure on R, $\bar{E}$ and E^o the closure and interior of the set E, $U(F, \varepsilon)$ the ε-neighbourhood of the set F, $f/_E$ the restriction of the function f to the domain E, and (x,y) the open interval from x to y where $x < y$ or $x > y$. For $h \in \mathcal{H}$, h^{-1} denotes the inverse of h. F_σ and G_δ denotes the collection of all sets of type F_σ and G_δ, respectively.

<u>Homeomorphic transformation of M_1-sets into M_5-sets</u>. In this section, by a measure we mean a nonnegative locally finite non-atomic Borel regular measure on R. A measure μ is called positive if $\mu(I) > 0$ for every open interval I.

Definition. Let $E \in F_\sigma \setminus \{\emptyset\}$ and let μ be a positive measure. We shall say that E belongs to class

M_0 if $E \cap I$ is infinite whenever I is a closed interval intersecting E (i.e., E is bilaterally dense-in-itself)

M_1 if $E \cap I$ is uncountable whenever I is a closed interval intersecting E (i.e., E is bilaterally c-dense-in-itself)

M_2^μ if $\mu(E \cap I) > 0$ whenever I is a closed interval intersecting E

M_5^μ if every point of E is a point of density of E relative to μ (i.e., $d_\mu(E,x) \equiv \lim_{y \to x} \frac{\mu(E \cap (x,y))}{\mu(x,y)} = 1$ for every $x \in E$).

The empty set is considered to belong to each of these classes.

Remark 1. It is easy to verify that the following assertions are valid for any positive measure μ.

(a) $M_0 \supset M_1 \supset M_2^\mu \supset M_5^\mu$. Any open set E belongs to M_5^μ.

(b) If $E \in M_1$, $h \in \mathcal{H}$, then $h(E) \in M_1$.

(c) Each of the defined classes is closed under the formation of countable unions. M_5^μ is closed under the formation of finite intersections, but none of the other classes is. To see this, put

$$A = (-1,0] \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{2n+1}, \frac{1}{2n}\right), \quad B = (-1,0] \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{2n}, \frac{1}{2n-1}\right).$$

Then A and B are in M_2^μ, but $A \cap B = (-1,0] \notin M_0$.

Lemma 1. Let $\{G_m\}_{m\in N} \subset M_1$. Then there exists a positive measure μ such that $\mu(-\infty,0) = \mu(0,\infty) = \infty$ and $\{G_m\}_{m\in N} \subset M_2^\mu$.

Proof. For any uncountable Borel set B there exists a finite measure γ such that $\gamma(B) > 0$ (see [7, p. 101]).

Let $\{I_n\}$ be a sequence of all closed intervals with rational endpoints. Put $M = \{(m,n): G_m \cap I_n \neq \emptyset\}$. If $(m,n) \in M$, then $G_m \cap I_n$ is uncountable, and we may find a measure $\gamma_{m,n}$ such that $\gamma_{m,n}(G_m \cap I_n) > 0$ and $\gamma_{m,n}(R) = 2^{-m-n}$. We set $\mu = \lambda + \sum_{(m,n)\in M} \gamma_{m,n}$.

Lemma 2. Suppose that

(a) $\varepsilon > 0$, η is a positive measure

(b) F is a nonempty compact nowhere dense set

(c) $E \in M_2^\eta$, $F \subset E$.

Then there exists a measure γ such that

(1) $\gamma(R \setminus (E \cap U(F,\varepsilon))) = 0$

(2) $\gamma(R) < \varepsilon$

(3) if $x \in F$, then $\lim_{y\to x} \frac{\eta((x,y)\setminus E)}{\gamma(x,y)} = 0$.

Proof. This is a corollary of [7, Lemma 2]: Under the hypotheses (a), (b), (c) there exists a Borel measurable nonnegative function g such that

(1') $\{x: g(x) \neq 0\} \subset (E \setminus F) \cap U(F,\varepsilon)$

(2') $\int_R g \, d\eta < \varepsilon$

(3') if $x \in F$, then $\lim_{y \to x} \eta((x,y) \setminus F).(\int_{(x,y)} g\, d\eta)^{-1} = 0$.

It suffices to put $\gamma(B) = \int_B g\, d\eta$ for every Borel set B.

Lemma 3. Suppose that

(a) $\varepsilon > 0$, μ and η are positive measures

(b) A and B are compact nowhere dense sets

(c) $E \in M_2^\eta$, $A \subset E$.

Then there exists a measure γ such that

(4) $\gamma(R \setminus E) = 0$

(5) $\gamma(R) < \varepsilon$

(6) if $x \in A \setminus B$, then $\lim_{y \to x} \frac{\eta((x,y) \setminus E)}{\gamma(x,y)} = 0$

(7) if $x \in B$ and $y \neq x$, then $\gamma(x,y) < \varepsilon(\mu(x,y))^2$.

Proof. If $A \setminus B = \emptyset$, put $\gamma = 0$. If $A \neq \emptyset$, $B = \emptyset$, apply Lemma 2 with $F = A$. Now, let $A \setminus B$ and B be nonempty. We can write $A \setminus B = \bigcup_{n=1}^{\infty} A_n$ where A_n are nonempty compact nowhere dense sets. For each n find $\delta_n > 0$ such that $B \cap \overline{U(A_n, \delta_n)} = \emptyset$ and put

$\varepsilon_n = \min\{\varepsilon 2^{-n}, \delta_n, \varepsilon 2^{-n}(\inf\{\mu(x,y) : x \in B, y \in \overline{U(A_n, \delta_n)}\})^2\}$.

Apply Lemma 2 with $\varepsilon = \varepsilon_n$, $F = A_n$ to obtain a measure γ_n with properties (1), (2), (3). Set $\gamma = \sum_{n=1}^{\infty} \gamma_n$.

Statements (4), (5) are easy consequences of (1), (2). Let $x \in A_m \subset A \setminus B$. Since $\gamma(x,y) \geqq \gamma_m(x,y)$ for any $y \neq x$, (6) follows from (3). To prove (7), let $x \in B$, $y \neq x$.

If $\gamma_n(x,y) > 0$ for some n, then (1) implies $(x,y) \cap U(A_n, \varepsilon_n) \neq \emptyset$. Pick $z \in (x,y) \cap U(A_n, \varepsilon_n)$. We have

$\gamma_n(x,y) \leqq \gamma_n(R) < \varepsilon_n \leqq \varepsilon 2^{-n}(\mu(x,z))^2 < \varepsilon 2^{-n}(\mu(x,y))^2$

which implies (7).

Theorem 1. Let $\{E_n\}_{n \in N}$ be a countable collection of sets. Then the following conditions are equivalent.

(i) $\bigcap_{j \in M} E_j \in M_1$ whenever $M \subset N$ is finite.

(ii) There exists a positive measure ν such that $\nu(-\infty,0) = \nu(0,\infty) = \infty$ and $\{E_n\}_{n \in N} \subset M_5^{\nu}$.

(iii) There exists $h \in \mathcal{H}$ such that $\{h(E_n)\}_{n \in N} \subset M_5^{\lambda}$.

Proof. (i)$\Rightarrow$(ii). By Lemma 1, we may find a positive measure μ such that $\mu(-\infty,0) = \mu(0,\infty) = \infty$ and $E_M \equiv \bigcap_{j \in M} E_j \in M_2^{\mu}$ whenever $M \subset N$ is finite.

We shall suppose that all the sets $H_n = E_n \setminus E_n^o$ are nonempty (if $H_n = \emptyset$, then E_n, being open, is in M_5^{ν} for any positive measure ν). Since for each n, $H_n \in F_\sigma$ and $H_n^o = \emptyset$, we may write $H_n = \bigcup_{k=n}^{\infty} H_n^k$ where $\{H_n^k\}_{k=n}^{\infty}$ is a sequence of compact nowhere dense sets such that $\emptyset \neq H_n^k \subset H_n^{k+1}$ for every $k \geqq n$.

Let P_n denote the collection of all nonempty subsets of $N_n = \{1,2,\dots,n\}$. For each $n \in N$ and each $M \in P_n$ put

$$\varepsilon_n = 2^{-(2n-1)}, \quad A_M^n = \bigcap_{j \in M} H_j^n, \quad B_M^n = \bigcup_{j \in N_n \setminus M} H_j^n$$

(if $M = N_n$, then $B_M^n = \emptyset$). Note that $A_M^n \subset E_M$.

We shall construct a sequence $\{\gamma_n\}_{n=0}^{\infty}$ of measures. We set $\gamma_0 = \mu$. Assume that $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ have already been defined. Put

$$\nu_{n-1} = \gamma_0 + \gamma_1 + \ldots + \gamma_{n-1} .$$

For each set $M \in P_n$ apply Lemma 3 with $\varepsilon = \varepsilon_n$, $\eta = \nu_{n-1}$, $A = A_M^n$, $B = B_M^n$, $E = E_M$ to obtain a measure γ_M^n such that

(4') $\gamma_M^n(R \setminus E_M) = 0$

(5') $\gamma_M^n(R) < \varepsilon_n$

(6') if $x \in A_M^n \setminus B_M^n$, then $\lim\limits_{y \to x} \dfrac{\nu_{n-1}((x,y) \setminus E_M)}{\gamma_M^n(x,y)} = 0$

(7') if $x \in B_M^n$, $y \neq x$, then $\gamma_M^n(x,y) < \varepsilon_n(\mu(x,y))^2$.

Put

$$\gamma_n = \sum_{M \in P_n} \gamma_M^n .$$

Having defined the sequence $\{\gamma_n\}_{n=0}^{\infty}$, set

$$\nu = \sum_{n=0}^{\infty} \gamma_n .$$

Fix $m \in N$, $x \in E_m$. We need to show that $d_\nu(E_m, x) = 1$. If $x \in E_m^o$, then it is obvious. Suppose that $x \in H_m$. Choose $i \geqq m$ so that $x \in H_m^i$. For $y \neq x$ we can write

$$\nu((x,y) \setminus E_m) = \nu_{i-1}((x,y) \setminus E_m) + \sum_{n=i}^{\infty} \gamma_n((x,y) \setminus E_m) .$$

Let $n \geqq i$, $M \in P_n$. If $m \in M$, then $E_M \subset E_m$, and (4') yields $\gamma_M^n(R \setminus E_m) = 0$. If $m \notin M$, then $x \in H_m^i \subset H_m^n \subset B_M^n$, hence $\gamma_M^n((x,y) \setminus E_m) \leqq \gamma_M^n(x,y) < \varepsilon_n(\mu(x,y))^2$, by (7').

Putting $P_n^* = \{M \in P_n : m \notin M\}$, we obtain $\gamma_n((x,y)\setminus E_m) =$

$$= \sum_{M\in P_n^*} \gamma_M^n((x,y)\setminus E_m) < 2^{n-1}\varepsilon_n(\mu(x,y))^2 = 2^{-n}(\mu(x,y))^2,$$

consequently

$$\sum_{n=i}^{\infty} \gamma_n((x,y)\setminus E_m) < (\mu(x,y))^2 \leqq \mu(x,y)\nu(x,y) .$$

Put $Q = \{j : x \in H_j^i\}$. Then $m \in Q$ and $x \in A_Q^i \setminus B_Q^i$. Since $E_Q \subset E_m$, we have

$$\nu_{i-1}((x,y)\setminus E_m) \leqq \nu_{i-1}((x,y)\setminus E_Q) \leqq \frac{\nu_{i-1}((x,y)\setminus E_Q)}{\gamma_Q^i(x,y)} \nu(x,y) .$$

Thus

$$\nu((x,y)\setminus E_m) \leqq \left(\frac{\nu_{i-1}((x,y)\setminus E_Q)}{\gamma_Q^i(x,y)} + \mu(x,y)\right) \nu(x,y)$$

and (6') proves that $d_\nu(E_m,x) = 1 - \lim\limits_{y\to x} \dfrac{\nu((x,y)\setminus E_m)}{\nu(x,y)} = 1$.

(ii) $\Rightarrow$ (iii). Define a function h by

$$h(x) = \begin{cases} \nu(0,x) & \text{if } x > 0 \\ -\nu(x,0) & \text{if } x < 0 \end{cases}, \quad h(0) = 0 .$$

Then $h \in \mathscr{H}$. Since $\nu(I) = \lambda(h(I))$ for every open interval I, we have $\nu(B) = \lambda(h(B))$ for every Borel set B.

Let $a = h(u)$, $u \in E_n = E$. Then

$$d_\lambda(h(E),a) = \lim_{b\to a} \frac{\lambda(h(E)\cap(a,b))}{\lambda(a,b)} =$$

$$= \lim_{v\to u} \frac{\lambda(h(E)\cap(h(u),h(v)))}{\lambda(h(u),h(v))} =$$

$$= \lim_{v\to u} \frac{\nu(E\cap(u,v))}{\nu(u,v)} = d_\nu(E,u) = 1 .$$

(iii) $\Rightarrow$ (i). We have $\bigcap\limits_{j\in M} E_j = h^{-1}(\bigcap\limits_{j\in M} h(E_j))$ for every finite set $M \subset N$. Apply Remark 1.

Classes $\mathfrak{M}_1$ and $\mathfrak{M}_5$. The only measure which will be used throughout the rest of this paper is the Lebesgue measure λ. So we shall write simply M_5, $d(E,x)$ instead of M_5^λ, $d_\lambda(E,x)$.

Definition. A function f is said to be in class $\mathfrak{M}_i$ $(i = 1,5)$ if every associated set of f is in M_i. (The associated sets of f are the sets $E^r(f) = \{x: f(x) < r\}$ and $E_r(f) = \{x: f(x) > r\}$ where $r \in R$.)

Remark 2. Referring to Remark 1, we immediately derive the following facts.

(a) $\mathfrak{M}_5 \subset \mathfrak{M}_1$.

(b) If $f \in \mathfrak{M}_1$, $h \in \mathcal{H}$, then $f \circ h \in \mathfrak{M}_1$.

(c) For a function f and $p,q \in R$ put

$$E_p^q(f) = E_p(f) \cap E^q(f) = \begin{cases} \{x: p < f(x) < q\} & \text{if } p < q \\ \emptyset & \text{if } p \geqq q \end{cases}.$$

If $f \in \mathfrak{M}_5$, then $E_p^q(f) \in M_5$ for all $p,q \in R$. Conversely, if $\{E_p^q(f): p,q \text{ rational}\} \subset M_5$, then $f \in \mathfrak{M}_5$.

If $f \in \mathfrak{M}_1$, it is not immediately clear whether or not $E_p^q(f) \in M_1$ for all $p,q \in R$ (see Remark 1.c). To give an affirmative answer to this question, we need the following

Lemma 4. Suppose that A and B are in $M_1 \setminus \{\emptyset\}$, $A \cup B = R$. Then $A \cap B \in M_1 \setminus \{\emptyset\}$.

Proof (cf. [3, Lemma 3.2.1]). By [8, Lemma 7], no open interval $I \subset R$ can be expressed as the union of two nonempty disjoint M_0-sets. Therefore $A \cap B \neq \emptyset$.

Let I be an open interval such that $A \cap B \cap \bar{I} \neq \emptyset$. Then $A \cap I$ and $B \cap I$ are uncountable. We show that $A \cap B \cap I$ is uncountable. If $I \subset A$ or $I \subset B$, then it is obvious. Suppose $I \setminus A \neq \emptyset$, $I \setminus B \neq \emptyset$. Since $R \setminus A \subset B$, $R \setminus A \in G_\delta$, $B \in F_\sigma$, there is a set $E \in F_\sigma \cap G_\delta$ such that $R \setminus A \subset E \subset B$. Using [8, Lemma 7] again, we obtain $\{E \cap I, I \setminus E\} \not\subset M_0$. Assume $E \cap I \notin M_0$, the other case being similar. Then there is an open interval $J \subset I$ with $E \cap \bar{J} \neq \emptyset$ and $E \cap J = \emptyset$. Since $E \subset B$, we have $B \cap \bar{J} \neq \emptyset$, so $B \cap J$ is uncountable. Furthermore, $J \subset R \setminus E \subset A$. Thus $A \cap B \cap I \supset A \cap B \cap J = B \cap J$ which implies the result.

<u>Corollary</u>. If $f \in \mathfrak{M}_1$, then $E_p^q(f) \in M_1$ for all $p, q \in R$. (Proof: If $p < q$, then $E_p(f) \cup E^q(f) = R$.)

<u>Definition</u>. A function f is said to be approximately continuous ($f \in \mathcal{A}$) if for each $x \in R$ there exists a measurable set E_x such that $x \in E_x$, $d(E_x, x) = 1$ and $f/_{E_x}$ is continuous at x.

A measurable set E is said to be D-open provided that $d(E,x) = 1$ for every $x \in E$.

<u>Remark 3</u>. The collection D of all D-open sets forms a topology (see e.g. [1, p. 20]). A function f belongs to $\mathcal{A}$ if and only if every associated set of f belongs to D (see e.g. [1, Chap. II, Theorem 5.6]). Thus $\mathcal{A}$ is exactly the class of all D-continuous functions. Consequently, if f, g, h are in $\mathcal{A}$, $h(x) \neq 0$ for all $x \in R$, then $f+g$, $f.g$, $\frac{f}{h}$ are in $\mathcal{A}$.

The class $\mathcal{DB}_1$ of all Darboux Baire 1 functions does not behave well with respect to the algebraic operations. To see this, put $f(x) = \sin \frac{1}{x}$, $g(x) = -\sin \frac{1}{x}$ if $x \neq 0$, $f(0) = g(0) = 1$. Then f, g are in $\mathcal{DB}_1$, but neither $f+g$ nor $f.g$ is. So, for any topology τ on R, $\mathcal{DB}_1$ cannot coincide with the class of all τ-continuous functions. Hence there is no topology τ on R for which $\mathcal{DB}_1$ is the class of all τ-continuous functions.

Remark 4. Since $M_5 = F_\sigma \cap D$ and since $\mathcal{A} \subset \mathcal{B}_1$ (see e.g. [1, Chap. II, Theorem 5.5]), we conclude that $\mathcal{M}_5 = \mathcal{A}$.

Zahorski proved that $\mathcal{M}_1 = \mathcal{DB}_1$ ([8, Theorem 1]).

Maximoff's theorem.

Theorem 2 (Maximoff [5], Preiss [7]). For any function f, the following conditions are equivalent.

(i) $f \in \mathcal{M}_1$.

(ii) There exists $h \in \mathcal{H}$ such that $f \circ h \in \mathcal{M}_5$.

Proof. (i) $\Rightarrow$ (ii). Put $S = \{E_p^q(f): p, q \text{ rational}\}$. By the corollary of Lemma 4, $S \subset M_1$. The intersection of any collection of finitely many sets from S belongs to S. Applying Theorem 1, we construct a homeomorphism $g \in \mathcal{H}$ such that $\{g(E): E \in S\} \subset M_5$. Put $h = g^{-1}$.

Let p, q be rational numbers. Then

$$E_p^q(f \circ h) = h^{-1}(E_p^q(f)) = g(E_p^q(f)) \in M_5.$$

Hence $f \circ h \in \mathcal{M}_5$, by Remark 2.c.

(ii) $\Rightarrow$ (i). This follows from the equality $f = (f \circ h) \circ h^{-1}$ and from Remark 2.a,b.

Zero sets and separation properties of $\mathfrak{M}_1$ and $\mathfrak{M}_5$. First we state two well-known lemmas of Zahorski concerning $\mathfrak{M}_5$-functions and their analogues for $\mathfrak{M}_1$-functions.

Theorem 3.i $(i = 1,5)$. If $E \in M_i$, then there exists an upper-semicontinuous function $f \in \mathfrak{M}_i$ such that

$$0 < f(x) \leqq 1 \text{ if } x \in E, \quad f(x) = 0 \text{ if } x \in R \setminus E.$$

Theorem 3.5 is due to Zahorski ([8, Lemma 11]). Theorem 3.1 is due to Agronsky (see Bruckner [1, p. 28-31]).

Proof of Theorem 3.1. Let $E \in M_1$. By Theorem 1 (or by [2]), there exists $h \in \mathcal{H}$ such that $h(E) \in M_5$. Using Theorem 3.5, we find a function $g \in \mathfrak{M}_5$ such that $0 < g(y) \leqq 1$ if $y \in h(E)$, $g(y) = 0$ if $y \in R \setminus h(E)$. We put $f = g \circ h$.

Theorem 4.i $(i = 1,5)$. Let H_1 and H_2 be nonempty disjoint sets such that $R \setminus H_1$ and $R \setminus H_2$ are in M_i. Then there exists a function $f \in \mathfrak{M}_i$ such that

$$f(x) = 0 \text{ if } x \in H_1, \quad f(x) = 1 \text{ if } x \in H_2,$$
$$0 < f(x) < 1 \text{ if } x \in R \setminus (H_1 \cup H_2).$$

Theorem 4.5 is due to Zahorski ([8, Lemma 12]). We give the original proof here in order to show that the same method fails to work in $\mathfrak{M}_1$ (see Remark 3):

Suppose that $R \setminus H_1$ and $R \setminus H_2$ are in M_5. By Theorem 3.5, there are functions $f_k \in \mathfrak{M}_5$ $(k = 1,2)$ such that $0 < f_k(x) \leqq 1$ if $x \in R \setminus H_k$ and $f_k(x) = 0$ if $x \in H_k$.

It suffices to put $f = \dfrac{f_1}{f_1 + f_2}$.

Proof of Theorem 4.1. Let $\{R\setminus H_1, R\setminus H_2\} \subset M_1$. Since $(R\setminus H_1)\cup(R\setminus H_2) = R$, we have $(R\setminus H_1)\cap(R\setminus H_2) \in M_1$, by Lemma 4. According to Theorem 1, there exists $h \in \mathcal{H}$ such that $\{R\setminus h(H_1), R\setminus h(H_2)\} \subset M_5$. Now take a function $g \in \mathfrak{M}_5$ from Theorem 4.5 applied to $h(H_1)$, $h(H_2)$ and put $f = g\circ h$.

Definition (Laczkovich [3]). Let $i \in \{1,5\}$.

A set H is said to be an $\mathfrak{M}_i$-zero set if there exists a function $f \in \mathfrak{M}_i$ such that $H = \{x: f(x) = 0\}$.

A set F is said to be $\mathfrak{M}_i$-closed if F coincides with the intersection of all $\mathfrak{M}_i$-zero sets which contain F.

A pair G_1, G_2 of disjoint sets is said to be separated by $\mathfrak{M}_i$ if there exists a function $f \in \mathfrak{M}_i$ such that

$$G_1 \subset \{x: f(x) = 0\}, \quad G_2 \subset \{x: f(x) = 1\}.$$

Remark 5. Let $i \in \{1,5\}$. If $f \in \mathfrak{M}_i$ and $r \in R$, then $R\setminus\{x: f(x) = r\} = E_r(f)\cup E^r(f) \in M_i$. Combining this fact with Theorem 3.i and Theorem 4.i, we obtain the following characterizations:

(a) A set H is an $\mathfrak{M}_i$-zero set if and only if $R\setminus H \in M_i$.

(b) A pair G_1, G_2 of disjoint sets is separated by $\mathfrak{M}_i$ if and only if there is a pair of disjoint sets H_1, H_2 such that $R\setminus H_1$, $R\setminus H_2$ are in M_i and $G_1 \subset H_1$, $G_2 \subset H_2$.

Remark 6. A set F is $\mathfrak{M}_5$-closed if and only if F is D-closed (see [3, p. 408]).

It remains to characterize all $\mathfrak{M}_1$-closed sets.

<u>Definition</u>. The class of sets C is defined by $A \in C$ if and only if $A \cap I$ contains a nonempty perfect set whenever I is a closed interval intersecting A.

<u>Remark 7</u>. If $A \in C$, then obviously A is bilaterally c-dense-in-itself. If E is a Borel set, then E is in C if and only if E is bilaterally c-dense-in-itself (apply the fact that any uncountable Borel set contains a nonempty perfect set). Thus $M_1 = F_\sigma \cap C$.

<u>Lemma 5</u>. Any set $A \in C$ contains a set E of type F_σ such that $E \cap I$ is uncountable whenever I is a closed interval intersecting A (i.e., E is bilaterally c-dense in A). (Observe that $E \in M_1$.)

<u>Proof</u>. Let $\{I_n\}$ be a sequence of all closed intervals with rational endpoints. Put $M = \{n \in N: A \cap I_n \neq \emptyset\}$. If $n \in M$, then $A \cap I_n$ contains a nonempty perfect set P_n. Define $E = \bigcup_{n \in M} P_n$.

<u>Lemma 6</u>. A set F is $\mathfrak{M}_1$-closed if and only if $R \setminus F \in C$.

<u>Proof</u>. Let F be an $\mathfrak{M}_1$-closed set, $F \neq R$. Choose a closed interval I intersecting $R \setminus F$, $x \in I \setminus F$. There is an $\mathfrak{M}_1$-zero set H such that $F \subset H$ and $x \notin H$. We have $x \in I \setminus H \subset I \setminus F$. Since $R \setminus H \in M_1$ by Remark 5.a, there is a nonempty perfect set P such that $P \subset I \setminus H \subset I \setminus F$. Hence $R \setminus F \in C$.

Suppose that $R \setminus F \in C$, $F \neq R$. Choose $x \in R \setminus F$. By Lemma 5, $R \setminus F$ contains a set $G \in F_\sigma$ which is bilaterally c-dense in $R \setminus F$. Put $E = G \cup \{x\}$. Then $E \in M_1$. Applying Theorem 3.1, we find a function $f \in \mathfrak{M}_1$ such that $f(y) > 0$ if $y \in E$, $f(y) = 0$ if $y \in R \setminus E$. So, $f(x) \neq 0$ and f vanishes on $F \subset R \setminus E$. This proves that F is $\mathfrak{M}_1$-closed.

Clearly, if $F = R$, then F is $\mathfrak{M}_1$-closed and $R \setminus F = \emptyset \in C$.

Remark 8. Let $i \in \{1,5\}$. If H is an $\mathfrak{M}_i$-closed set of type G_δ, then $R \setminus H \in M_i$ (see Lemma 6 and Remark 6). So, Theorem 4.i implies that any pair of disjoint $\mathfrak{M}_i$-closed sets of type G_δ is separated by $\mathfrak{M}_i$. This fact with $i = 1$ is due to Laczkovich (see [3, Theorem 3.2.2]).

Extension theorems for $\mathfrak{M}_5$ and $\mathfrak{M}_1$. This section is devoted to modifications of the classical Tietze's theorem.

Theorem 5.5 (Petruska, Laczkovich [6, Theorem 3.2]). For any set H, the following conditions are equivalent.

(i) $\lambda(H) = 0$.

(ii) For each $g \in \mathcal{B}_1$ there exists $f \in \mathfrak{M}_5$ such that $f/_H = g/_H$.

Theorem 6.5 (Lukeš [4, Theorem 4]). Let F be a D-closed set and let $g \in \mathcal{B}_1$. Then the following conditions are equivalent.

(i) $g/_F$ is D-continuous on F.

(ii) There exists a function $f \in \mathfrak{M}_5$ such that $f/_F = g/_F$.

Remark 9. Let g be a function, F a set, $p,q,r \in R$. We introduce the following notations:

$$E^r(g,F) = E^r(g) \cup (R \setminus F), \quad E_r(g,F) = E_r(g) \cup (R \setminus F),$$

$$E_p^q(g,F) = E_p^q(g) \cup (R \setminus F).$$

Let F be D-closed. Then $g/_F$ is D-continuous on F if and only if $E^r(g,F) \in D$ and $E_r(g,F) \in D$ for all $r \in R$.

Remark 10. Suppose that $\lambda(H) = 0$ and $g \in \mathcal{B}_1$. Then H is D-closed and $g/_H$ is D-continuous on H. Thus the implication (i) $\Rightarrow$ (ii) of Theorem 5.5 is a corollary of Theorem 6.5.

Theorem 6.1 Let F be an $\mathfrak{M}_1$-closed set (i.e., $R \setminus F \in C$), and let $g \in \mathcal{B}_1$. Then the following conditions are equivalent.

(i) $E^r(g,F) \in C$ and $E_r(g,F) \in C$ for all $r \in R$.

(ii) There exists a function $f \in \mathfrak{M}_1$ such that $f/_F = g/_F$.

Proof. (i) $\Rightarrow$ (ii). By Lemma 5, there is a set H such that $F \subset H$, $R \setminus H \in F_\sigma$ and $R \setminus H$ is bilaterally c-dense in $R \setminus F$ (hence $R \setminus H \in M_1$). It is easy to show that $E^r(g,H) \in M_1$ and $E_r(g,H) \in M_1$ for all $r \in R$. Hence $E_p^q(g,H) \in M_1$ for all $p,q \in R$, $p < q$ (apply Lemma 4). If $p \geqq q$, then $E_p^q(g,H) = R \setminus H \in M_1$.

Put $S = \{E_p^q(g,H) : p,q \text{ rational}\}$. Obviously, S is closed under the formation of finite intersections. Using Theorem 1, we construct a homeomorphism $h \in \mathcal{H}$ such that $\{h(E) : E \in S\} \subset M_5$. Since $R \setminus h(H) \in M_5$, $h(H)$ is D-closed.

Define $g^* = g \circ h^{-1}$. Clearly, $g^* \in \mathcal{B}_1$. For all rational p, q we have $E_p^q(g^*, h(H)) = h(E_p^q(g,H)) \in M_5$. Hence $g^*/_{h(H)}$ is D-continuous on $h(H)$.

According to Theorem 6.5, there exists $f^* \in \mathfrak{M}_5$ such that $f^*/_{h(H)} = g^*/_{h(H)}$. Put $f = f^* \circ h$. Then $f \in \mathfrak{M}_1$ and $f/_H = g/_H$, so a fortiori $f/_F = g/_F$.

(ii) $\Rightarrow$ (i). This follows from the equalities

$$E^r(g,F) = E^r(g) \cup (R \setminus F) = E^r(f) \cup (R \setminus F),$$

$$E_r(g,F) = E_r(g) \cup (R \setminus F) = E_r(f) \cup (R \setminus F)$$

and from the fact that $E^r(f)$, $E_r(f)$, $R \setminus F$ are in C.

Theorem 5.1 For any set H, the following conditions are equivalent.

(i) For any interval I, $I \setminus H$ contains a nonempty perfect set.

(ii) For each $g \in \mathcal{B}_1$ there exists $f \in \mathfrak{M}_1$ such that $f/_H = g/_H$.

Proof. (i) $\Rightarrow$ (ii). It is clear that $E^r(g,H) \in C$ and $E_r(g,H) \in C$ for any $g \in \mathcal{B}_1$ and $r \in R$. The result follows from Theorem 6.1.

not(i) $\Rightarrow$ not(ii). Suppose that there is an interval I such that $I \setminus H$ does not contain any nonempty perfect set. Choose $x \in H \cap I$. Put $g(x) = 1$, $g(y) = 0$ for all $y \neq x$. Obviously, $g \in \mathcal{B}_1$. Assume that there exists $f \in \mathfrak{M}_1$ such that $f/_H = g/_H$. Then $E \equiv E_0(f) \in M_1$. Since $x \in E \cap I$, $E \cap I$ is uncountable. Therefore $(E \cap I) \setminus \{x\}$ contains some nonempty perfect set P. But $(E \cap I) \setminus \{x\} \subset I \setminus H$, hence $P \subset I \setminus H$ - a contradiction.

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