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ON SEMICONTINUITY POINTS.

Let f be a real function of one variable. Z.Grande gave in [1] some properties of the set of all points in which f is upper semicontinuous. In this paper we shall study the relatioship between the sets $S(f)$ and $S^{\uparrow}(f)$ where $S(f)$ / resp. $S^{\uparrow}(f)$ / is the set of all points at which f is upper / resp. lower / semicontinuous.

We use the notation introduced in [1]:

$$C(f)=\{x: \quad f(x)=\limsup_{t\to x} f(t)=\liminf_{t\to x} f(t)\},$$

$$S(f)=\{x: \quad f(x)\geq \limsup_{t\to x} f(t)\},$$

$$S^{\uparrow}(f)=\{x: \quad f(x)\leq \liminf_{t\to x} f(t)\},$$

$$T(f)=\{x: \quad f(x)> \limsup_{t\to x} f(t)\},$$

$$T^{\uparrow}(f)=\{x: \quad f(x)< \liminf_{t\to x} f(t)\}.$$

$A^{\odot}$ denotes the set of the points of condensation of A.

Let us recall some useful facts /see for example [1]/ :

i/ the set $T(f)$ is countable,

ii/ $C(f)$ is a G_δ set,

iii/ $C(f)$ is dense in the set Int $S(f)$. Similarly,

i^1/ the set $T^1(f)$ is countable,

iii^1/ $C(f)$ is dense in the set Int $S^1(f)$.

We are going to prove the following theorem:

THEOREM. If A, A^1, B, C, C^1 are subsets of R such that:

i/ $A \cap A^1 = B$,

ii/ B is dense in the set $\mathrm{Int}(A) \cup \mathrm{Int}(A^1)$,

iii/ B is a G_δ set,

iv/ $C \subseteq A - B$ and $C^1 \subseteq A^1 - B$,

v/ $C \cup C^1$ is countable,

then there exists a function $f: R \to R$ such that

$A = S(f)$, $A^1 = S^1(f)$, $B = C(f)$, $C = T(f)$ and $C^1 = T^1(f)$.

P r o o f :   Let $E = Cl(B)$. Then $E - B$ is a F_σ set and, moreover, it is of the first category. So, $E - B = \bigcup_{n\in N} F_n$, where F_n is closed for $n \in N$ and $F_i \cap F_j = \emptyset$ for $i \neq j$ / cf[3]/.

By the Cantor - Bendixon theorem, there is a partition of the set F_n, say $F_n = H_n \cup G_n$, such that $G_n = F_n^{\circledcirc}$ and H_n is countable for $n \in N$.

I. In the first step we shall construct a function $g: R \to R$ such that $C(g) = R - \bigcup_{n\in N} H_n$, $S\,g = (R - \bigcup_{n\in N} H_n) \cup (A \cap \bigcup_{n\in N} H_n)$, $S^1(g) = A^1 \cap \bigcup_{n\in N} H_n$ and $T(g) = T^1(g) = \emptyset$.

Let $(a_m)_{m\in N}$ be an enumeration of $\bigcup_{n\in N} H_n$ and $(b_m)_{m\in N}$ be a sequence of positive real numbers such that $\sum_{n\in N} b_n = 1$.

The function g is defined as follows:

$$g(x) = \begin{cases} \sum_{\{i:\ a_i \leqslant x\}} b_i & \text{for } x \in (R - \bigcup_{n\in N} H_n) \cup (\bigcup_{n\in N} H_n \cap A), \\ \sum_{\{i:\ a_i < x\}} b_i & \text{for } x \in A^1 \cap \bigcup_{n\in N} H_n, \\ \sum_{\{i:\ a_i < x\}} b_i + \frac{1}{2} b_j & \text{for } x \in \bigcup_{n\in N} H_n - (A \cup A^1) \text{ and } x = a_j; \end{cases}$$

It is easy to show that g satisfies the above conditions.

II. In the second step we shall construct a function $h: R \to R$ such that $S(h) = (R - (E - B)) \cup A$, $S^1(h) = (R - (E - B)) \cup A^1$, $C(h) = R - (E - B)$ and $T(h) = T^1(h) = \emptyset$.

It is clear that $G_n = (G_n \cap A) \cup (G_n \cap A^1) \cup (G_n - (A \cup A^1))$.

By the Cantor - Bendixon theorem we have $G_n \cap A = K_n \cup L_n$, where $K_n \cap L_n = \emptyset$, $K_n = (G_n \cap A)^{\circledcirc}$ and L_n is countable.

Similarly, $G_n \cap A^1 = M_n \cup N_n$ and $G_n - (A \cup A^1) = S_n \cup T_n$, where $M_n \cap N_n = \emptyset$, $S_n \cap T_n = \emptyset$ T_n and N_n are countable,

$M_n = (G_n \cap A^1)^{\circledcirc}$, and $S_n = (G_n - (A \cup A^1))^{\circledcirc}$.

For each n the sets K_n, M_n, S_n are eiter uncountable or empty.

For each nonempty set S_n let us define the function

$$i_n : S_n \to \left(-\frac{1}{2n}, \frac{1}{2n}\right)$$

Let the family of sets $\mathcal{B}$ be a countable basis of S_n and $\mathcal{C}$ be a countable basis of $\left(-\frac{1}{2n}, \frac{1}{2n}\right)$. Then the family $\mathcal{B} \times \mathcal{C}$ is the countable basis of $S_n \times \left(-\frac{1}{2n}, \frac{1}{2n}\right)$.

Now let $(B_n \times C_n)_{n \in N}$ be an enumeration of $\mathcal{B} \times \mathcal{C}$, (x_{α}) be a transfinite enumeration of S_n and (y_γ) be a transfinite enumeration of $\left(-\frac{1}{2n}, \frac{1}{2n}\right)$.

We shall construct inductively the function $i_n^1 : \mathcal{B} \times \mathcal{C} \to S_n$:

$$i_n^1(B_m \times C_m) = \min_{\gamma}\left\{ x_\gamma \in S_n : \quad x_\gamma \in B_m - \bigcup_{k<m} i_n^1(B_k \times C_k)\right\}.$$

Define:

$$i_n(x) = \begin{cases} \min_{\zeta}\left\{ y_\zeta \in \left(-\frac{1}{2n}, \frac{1}{2n}\right) : \quad y_\zeta \in C_m\right\} & \text{for } x = i_n^1\, B_m \times C_m, \\ 0 & \text{otherwise.} \end{cases}$$

The function i_n has the property that for each $x \in Cl(S_n)$

$$\liminf_{t \to x} i_n(t) = -\frac{1}{2n} \quad \text{and} \quad \limsup_{t \to x} i_n(t) = \frac{1}{2n}.$$

We define the function $h : R \to R$.

$$h(x) = \begin{cases} g(x) + \frac{1}{n} & \text{for } x \in K_n, \\ g(x) - \frac{1}{n} & \text{for } x \in M_n, \\ g(x) + i_n(x) & \text{for } x \in S_n, \\ g(x) + \frac{1}{2n} & \text{for } x \in [L_n \cap Cl(S_n)] \cup [T_n \cap (R - Cl(M_n)) \cap Cl(K_n)], \\ g(x) - \frac{1}{2n} & \text{for } x \in [N_n \cap Cl(S_n)] \cup [T_n \cap Cl(M_n)], \\ g(x) & \text{for } x \in [L_n \cap ClM_n - ClS_n] \cup [N_n \cap ClK_n - ClS_n] \cup [R - (E - B)] \cup \bigcup_{k \in N} H_k. \end{cases}$$

Let us first prove that R is the domain of h. It is enough to show that $h(x)$ is defined for each $x \in G_n$.

If $x \in G_n$ then $x \in G_n \cap A$ or $x \in G_n \cap A^1$ or $x \in G_n - (A \cup A^1)$ but

a/ $G_n \cap A^1 = M_n \cup [N_n \cap Cl(S_n)] \cup [N_n \cap Cl(K_n) - Cl(S_n)]$,

b/ $G_n \cap A = K_n \cup [L_n \cap Cl(S_n)] \cup [L_n \cap Cl(M_n) - Cl(S_n)]$,

c/ $G_n - (A \cup A^1) = S_n \cup [T_n \cap Cl(M_n)] \cup [T_n \cap Cl(K_n) - Cl(M_n)]$.

In fact, if $x \in G_n \cap A$ and $x \notin K_n$, then $x \in L_n$. Since K_n is closed in $G_n \cap A$, there exists an open set U such that $x \in U$ and $U \cap K_n = \emptyset$. Assume that $x \notin Cl\ S_n$. Then there exists an open set V such that $x \in V$, $V \subseteq U$ and $V \cap S_n = \emptyset$, but for each open $V_1 \subseteq V$, $V_1 \cap G_n$ is uncountable, so $V_1 \cap M_n \neq \emptyset$ and $x \in L_n \cap Cl(M_n) - Cl(S_n)$.

The same arguments work in the cases a/ and c/.

Since for $n \in N$ the sets K_n, L_n, M_n, N_n, S_n, T_n are pairwise disjoint, the function h is well defined.

Let us now show that h satisfies the required conditions.

a/ Since $R - E$ is open, $h | R - E = g | R - E$, and $R - E \subseteq C(g)$, it follows that $R - E \subseteq C(h)$.

b/ If $x \in B$ and (x_n) is a sequence of elements of $(R - E) \cup B$

and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} h(x_n) = h(x)$.

If (x_n) is a sequence of elements of $E - B$, then there are sets $F_{j(n)}$

such that $x_n \in F_{j(n)}$ and $\lim_{n\to\infty} j(n) = \infty$ / because for every k $x \notin F_k$ /.

Then, $g(x_n) - \frac{1}{j(n)} \leqslant h(x_n) \leqslant g(x_n) + \frac{1}{j(n)}$, so $\lim_{n\to\infty} h(x_n) = h(x)$.

Therefore for every sequence (x_n), if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} h(x_n) = h(x)$

Thus $B \subseteq C(h)$.

c/ If $x \in H_n \cap A$ and (x_k) is a sequence of elements of $(R - E) \cup B$

such that $\lim_{k\to\infty} x_k = x$, then $\limsup_{k\to\infty} h(x_k) \leqslant h(x)$.

Assume that (x_k) is a sequence of elements of $E - B$ and $\lim_{k\to\infty} x_k = x$.

The following two cases may happen:

1/ there exists a sequence $j(k)_{k \in N}$ such that $\lim_{k\to\infty} j(k) = \infty$

and $x_k \in F_{j(k)}$, or

2/ there exist a subsequence (x_{k_m}) of (x_k) such that $x_{k_m} \in H_n$

for each $m \in N$.

It is easy to show that in case 1/ $\limsup_{k\to\infty} h(x_k) \leq h(x)$ and in case 2/ $\limsup_{k\to\infty} h(x_{k_m}) \leq h(x)$. Therefore $\limsup_{t\to x} h(t) \leq h(x)$.

Let (x_k) be a sequence such that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} g(x_k) = \limsup_{t\to x} g(t)$ and $\limsup_{t\to x} g(t) = g(x)$. Then $\lim_{k\to\infty} h(x_k) = \lim_{k\to\infty} g(x_k) = h(x)$ and $\limsup_{t\to x} h(t) = h(x)$.

Let (x_k) be a sequence such that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} g(x_k) = \liminf_{t\to x} g(t)$ / and $\liminf_{t\to x} g(t) < g(x)$ /. Then $\lim_{k\to\infty} h(x_k) = \lim_{k\to\infty} g(x_k) < h(x)$ and $\liminf_{t\to x} h(t) < h(x)$.

Thus $\bigcup_{n\in N} H_n \cap A \subseteq S(h) - (C(h) \cup T(h))$. A similar reasoning shows that $\bigcup_{n\in N} H_n \cap A^1 \subseteq S^1(h) - (C(h) \cup T^1(h))$.

d/ Let $x \in H_n - (A \cup A^1)$ and (x_k) be a sequence such that $\lim_{k\to\infty} g(x_k) = \limsup_{t\to x} g(t) > g(x) = h(x)$. Then $\limsup_{t\to x} h(t) \geq \lim_{k\to\infty} h(x_k)$ and $\lim_{k\to\infty} h(x_k) = \lim_{k\to\infty} g(x_k) > g(x) = h(x)$.

Similarly, if (y_k) is a sequence such that $\lim_{k\to\infty} g(y_k) = \liminf_{t\to x} g(t)$ / $\liminf_{t\to x} g(t) < g(x) = h(x)$/, then $\liminf_{t\to x} h(t) \leq \lim_{k\to\infty} g(y_k) < h(x)$ and $\bigcup_{n\in N} H_n - (A \cup A^1) \subseteq R - (S(h) \cup S^1(h))$.

e/ If $x \in K_n$, then there exists a sequence (x_m) such that for $m \in N$

$x_m \in K_n$ and $\lim_{m\to\infty} x_m = x$. Then $\limsup_{t\to x} h(t) = g(x) + \frac{1}{n} = h(x)$.

Since the set F_n is nowhere dense, there exists a sequence (x_m) such

that $\lim_{m\to\infty} x_m = x$ and $x_m \in R - (E - B)$ for $m \in N$. Therefore

$\liminf_{t\to x} h(t) \leq \lim_{m\to\infty} g(x_m) \leq g(x) < h(x)$ and $K_n \subseteq S(h) - (C(h) \cup T(h))$.

Similarly, if $x \in M_n$, then $\liminf_{t\to x} h(t) = g(x) - \frac{1}{n} = h(x)$ and

$\limsup_{t\to x} h(t) \geq g(x) > h(x)$. Thus $M_n \subseteq S^1(h) - (C(h) \cup T^1(h))$.

f/ If $x \in L_n \cap Cl(S_n)$, then there exists a sequence (x_m) in S_n such

that for each m $h(x_m) = g(x_m) + i_n(x_m) > g(x_m) + \frac{1}{2n} - \frac{1}{m}$.

Hence, $\limsup_{t\to x} h(t) \geq g(x) + \frac{1}{2n} = h(x)$.

Since $x \notin Cl(K_n)$, $\limsup_{t\to x} h(t) < g(x) + \frac{1}{n}$. Therefore, $\limsup_{t\to x} h(t) = h(x)$

and $\liminf_{t\to x} h(t) \leq g(x) < h(x)$. Thus $L_n \cap Cl(S_n) \subseteq S(h) - (C(h) \cup T(h))$.

Similarly, if $x \in N_n \cap Cl(S_n)$, then $\liminf_{t\to x} h(t) = h(x)$ and

$\limsup_{t\to x} h(t) \geq g(x) > h(x)$, so $N_n \cap Cl(S_n) \subseteq S^1(h) - (C(h) \cup T^1(h))$.

g/ If $x \in T_n \cap Cl(M_n)$, then $\liminf_{t\to x} h(t) = g(x) - \frac{1}{n} < h(x)$ and

$\limsup_{t\to x} h(t) \geq g(x) > h(x)$, so $T_n \cap Cl(M_n) \subseteq R - (S(h) \cup S^1(h))$.

Similarly, if $x \in T_n \cap Cl(K_n) - Cl(M_n)$, then

$\limsup_{t \to x} h(t) = g(x) + \frac{1}{n} > h(x)$ and $\liminf_{t \to x} h(t) \leqslant g(x) < h(x)$, so

$$T_n \cap Cl(K_n) - Cl(M_n) \subseteq R - (S(h) \cup S^1(h)).$$

h/ If $x \in L_n \cap Cl(M_n) - Cl(S_n)$, then

$\liminf_{t \to x} h(t) = g(x) - \frac{1}{n} < h(x)$ and $\limsup_{t \to x} h(t) \geqslant g(x) = h(x)$.

Since $x \notin Cl(K_n) \cup Cl(S_n)$, $\limsup_{t \to x} h(t) < g(x) + \frac{1}{2n}$. Hence,

$$L_n \cap Cl(M_n) - Cl(S_n) \subseteq S(h) - (C(h) \cup T(h)).$$

Similarly, if $x \in N_n \cap Cl(K_n) - Cl(S_n)$, then

$\limsup_{t \to x} h(t) = g(x) + \frac{1}{n} > h(x) = \liminf_{t \to x} h(t)$, so

$$N_n \cap Cl(K_n) - Cl(S_n) = S^1(h) - (C(h) \cup T^1(h)).$$

We have just proved that the function h has the following

properties:

$$S(h) = (R - (E - B)) \cup (A \cap \bigcup_{n \in N} H_n) \cup (A \cap \bigcup_{n \in N} G_n) = (R - (E - B)) \cup A,$$

$S^1(h) = (R - (E - B)) \cup A^1$, $\quad C(h) = R - (E - B)$ and

$$T(h) = T^1(h) = \emptyset.$$

III. Let (c_n , d_n) be a sequence of pairwise disjoint, open intervals such that $R - E = \bigcup_{n\in N}(c_n , d_n)$. For each $n \in N$,

$$(c_n , d_n) = [(c_n , d_n) \cap A] \cup [(c_n , d_n) \cap A^1] \cup [(c_n , d_n) - (A \cup A^1)].$$

Let $(c_n , d_n) \cap A = A_n \cup B_n$, where $A_n = ((c_n , d_n) \cap A)^{\circ}$ and B_n is countable. Similarly,

$$(c_n , d_n) \cap A^1 = C_n \cup D_n,\ C_n = ((c_n , d_n) \cap A^1)^{\circ},\ D_n \text{ is countable},$$

$$(c_n , d_n) - (A \cup A^1) = I_n \cup J_n,\ I_n = ((c_n , d_n) - (A \cup A^1))^{\circ},\ J_n \text{ is countable.}$$

In the third step we shall construct the function $m: R \to R$ such that $C(m) = B$, $T(m) = T^1(m) = \emptyset$, $S(m) = A \cup \bigcup_{n\in N} J_n \cap Cl(C_n) - Cl(A_n)$

and $S^1(m) = A^1 \cup \bigcup_{n\in N} J_n \cap Cl(A_n) - Cl(C_n)$,

For each $n \in N$ we shall define the function $l_n: I_n \to (-\frac{1}{2} , 0) \cup (0 , \frac{1}{2})$ such that for every $x \in Cl(I_n)$, $\limsup_{t\to x} l_n(t) = \frac{1}{2} = - \liminf_{t\to x} l_n(t)$. The definition of l_n is very much similar to the definition of the functions i_n from section II.

Let

$$m(x)=\begin{cases} h(x)+\operatorname{dist}(x,E) & \text{for } x\in\bigcup_{n\in N}A_n, \\ h(x)-\operatorname{dist}(x,E) & \text{for } x\in\bigcup_{n\in N}C_n, \\ h(x)+\operatorname{dist}(x,E)\cdot l_n(x) & \text{for } x\in\bigcup_{n\in N}I_n, \\ h(x)+\frac{1}{2}\operatorname{dist}(x,E) & \text{for } x\in\bigcup_{n\in N}B_n\cap Cl(I_n), \\ h(x)-\frac{1}{2}\operatorname{dist}(x,E) & \text{for } x\in\bigcup_{n\in N}D_n\cap Cl(I_n), \\ h(x) & \text{otherwise.}\end{cases}$$

Notice that

$$(c_n, d_n)\cap\{x\in R : m(x)=h(x)\}=[J_n\cap Cl(A_n)\cap Cl(C_n)]\cup$$
$$\cup[J_n\cap Cl(A_n)-Cl(I_n)]\cup[D_n\cap Cl(A_n)-Cl(I_n)]\cup$$
$$\cup[B_n\cap Cl(C_n)-Cl(I_n)]\cup[J_n\cap Cl(C_n)-Cl(A_n)].$$

This is a direct consequence of the following:

i/ $(c_n, d_n)=A_n\cup C_n\cup I_n\cup B_n\cup D_n\cup J_n$,

ii/ the sets A_n, C_n, I_n, B_n, D_n, J_n are pairwise disjoint,

iii/ $B_n\subseteq Cl(C_n)\cup Cl(I_n)$, $D_n\subseteq Cl(A_n)\cup Cl(I_n)$, $J_n\subseteq Cl(A_n)\cup Cl(C_n)$.

The case iii/ follows from the fact that if e.g. $x\in B_n$, then there exists an open set $U\subseteq(c_n, d_n)$ such that $x\in U$ and $U\cap A_n=\emptyset$

/since A_n is closed in $(c_n, d_n) \cap A$ /. Since U is uncountable and $B_n \cup D_n \cup J_n$ is countable, $(C_n \cup I_n) \cap V \neq \emptyset$, where V is any open subset of U such that $x \in V$.

a/ If $x \in E$ and (x_n) is a sequence such that $\lim_{n\to\infty} x_n = x$, then for every $n \in N$, $h(x_n) - \mathrm{dist}(x_n, E) \leq m(x_n) \leq h(x_n) + \mathrm{dist}(x_n, E)$. Then

$\limsup_{t\to x} m(t) = \limsup_{t\to x} h(t)$, $\liminf_{t\to x} m(t) = \liminf_{t\to x} h(t)$ and

$m(x) = h(x)$. Hence,

$$E \cap S(m) = E \cap S(h) = E \cap A, \qquad E \cap S^1(m) = E \cap S^1(h) = E \cap A^1,$$

$$E \cap C(m) = E \cap C(h) = E \cap B = B \quad \text{and} \quad E \cap T(m) = E \cap T^1(m) = \emptyset.$$

b/ If $x \in A_n$, then $\limsup_{t\to x} m(t) = \limsup_{t\to x} h(t) + \mathrm{dist}(t, E) = m(x)$.

Since $A_n \subseteq A$, $(c_n, d_n) \cap B = \emptyset$ and B is dense in $\mathrm{Int}(A)$, there exists a sequence (x_k) such that $\lim_{k\to\infty} x_k = x$ and $x_k \notin A_n$ / for $k \in N$ /. Then

$$m(x_k) \leq h(x_k) + \frac{1}{2}\,\mathrm{dist}(x_k, E) \quad \text{and}$$

$$\liminf_{t\to x} m(t) \leq \liminf_{k\to\infty} m(x_k) \leq h(x) + \frac{1}{2}\,\mathrm{dist}(x, E) < m(x). \text{ Hence,}$$

$$\bigcup_{n\in N} A_n \subseteq S(m) - (C(m) \cup T(m)). \qquad \text{Similarly,}$$

$$\bigcup_{n\in N} C_n \subseteq S^1(m) - (C(m) \cup T^1(m)).$$

c/ If $x \in I_n$, then $\limsup_{t \to x} m(t) \geqslant h(x) + \frac{1}{2}\operatorname{dist}(x,E) > m(x)$ and $m(x) > h(x) - \frac{1}{2}\operatorname{dist}(x,E) \geqslant \liminf_{t \to x} m(t)$. Therefore

$$\bigcup_{n \in N} I_n \subseteq R - \left(S(m) \cup S^1(m)\right).$$

d/ If $x \in B_n \cap Cl(I_n)$, then $\liminf_{t \to x} m(t) \leqslant h(x) - \frac{1}{2}\operatorname{dist}(x,E) < m(x)$ and $\limsup_{t \to x} m(t) = h(x) + \frac{1}{2}\operatorname{dist}(x,E) = m(x)$. Hence,

$$\bigcup_{n \in N} B_n \cap Cl(I_n) \subseteq S(m) - \left(C(m) \cup T(m)\right).$$ Similarly,

$$\bigcup_{n \in N} D_n \cap Cl(I_n) \subseteq S^1(m) - \left(C(m) \cup T^1(m)\right).$$

e/ If $x \in J_n \cap Cl(A_n) \cap Cl(C_n)$, then there exist sequences (x_k), (y_k) such that $x_k \in A_n$, $y_k \in C_n$ / for $k \in N$ / and $\lim_{k \to \infty} x_k = \lim_{k \to \infty} y_k = x$. Then $\limsup_{t \to x} m(t) \geqslant \lim_{k \to \infty} m(x_k) = h(x) + \operatorname{dist}(x,E) > m(x)$ and

$$m(x) > h(x) - \operatorname{dist}(x,E) = \lim_{k \to \infty} m(y_k) \geqslant \liminf_{t \to x} m(t).$$ Hence,

$$\bigcup_{n \in N} J_n \cap Cl(A_n) \cap Cl(C_n) \subseteq R - \left(S(m) \cup S^1(m)\right).$$

f/ If $x \in J_n \cap Cl(A_n) - Cl(C_n)$, then there exists an open set U such that $x \in U$, $U \cap I_n = \emptyset$ and $U \cap C_n = \emptyset$. Because the set A_n is boundary, $x \in Cl\left(J_n \cup B_n \cup D_n\right)$. But $U \cap B_n = \emptyset$. In fact, suppose that there exists $y \in U \cap B_n$. Then there exists an open set $V \subseteq U$

such that $y \in V$ and $V \cap A_n = \emptyset$. Then $V \cap (C_n \cup I_n \cup A_n) = \emptyset$,

a contradiction. Then

$$U = (U \cap A_n) \cup [U \cap J_n \cap Cl(A_n) - Cl(C_n)] \cup [U \cap D_n \cap Cl(A_n) - Cl(I_n)].$$

Therefore $\limsup_{t \to x} m(t) = h(x) + dist(x,E) > m(x) = h(x) = \liminf_{t \to x} m(t)$,

and so

$$\bigcup_{n \in N} J_n \cap Cl(A_n) - Cl(C_n) \subseteq S^1(m) - (C(m) \cup T^1(m)).$$ Similarly,

$$\bigcup_{n \in N} D_n \cap Cl(A_n) - Cl(I_n) \subseteq S^1(m) - (C(m) \cup T^1(m)),$$

$$\bigcup_{n \in N} B_n \cap Cl(C_n) - Cl(I_n) \subseteq S(m) - (C(m) \cup T(m)),$$

$$\bigcup_{n \in N} J_n \cap Cl(C_n) - Cl(A_n) \subseteq S(m) - (C(m) \cup T(m)).$$

Notice that for $x \in J_n \cap Cl(C_n) - Cl(A_n)$, $\liminf_{t \to x} m(t) = h(x) - dist(x,E)$

and $\limsup_{t \to x} m(t) = h(x)$. Hence,

$$S(m) = (A \cap E) \cup \bigcup_{n \in N} (A_n \cup [B_n \cap Cl(I_n)] \cup [B_n \cap Cl(C_n) - Cl(I_n)] \cup$$

$$\cup [J_n \cap Cl(C_n) - Cl(A_n)]) = A \cup \bigcup_{n \in N} J_n \cap Cl(C_n) - Cl(A_n),$$

$$S^1(m) = (A^1 \cap E) \cup \bigcup_{n \in N} (C_n \cup [D_n \cap Cl(I_n)] \cup [D_n \cap Cl(A_n) - Cl(I_n)] \cup$$

$$\cup [J_n \cap Cl(A_n) - Cl(C_n)]) = A^1 \cup \bigcup_{n \in N} J_n \cap Cl(A_n) - Cl(C_n),$$

$$C(m) = B \quad \text{and} \quad T(m) = T^1(m) = \emptyset.$$

IV. In the next step we define a function $p:R \to R$ such that

$$S(p)=A, \quad S^1(p)=A^1, \quad C(p)=B \text{ and } T(p)=T^1(p)=\emptyset.$$

The set $\bigcup_{n\in N} J_n-(Cl(C_n)\cap Cl(A_n))$ is countable. Let $(c_n)_{n\in N}$ be an enumeration of this set and $(d_n)_{n\in N}$ be a sequence of positive real numbers such that $\sum_{n N} b_n=\frac{1}{2}$.

Let us define the function $p:R \to R$ as follows:

$$p(x)=\begin{cases} m(x)+\operatorname{dist}(x,E)\sum_{\{n:\ c_n< x\}} d_n & \text{for } x\in \bigcup_{n\in N} J_n\cap Cl(C_n)-Cl(A \\ m(x)+\operatorname{dist}(x,E)\sum_{\{n:\ c_n\leq x\}} d_n & \text{otherwise.}\end{cases}$$

a/ If $x\notin \bigcup_{n\in N} J_n-(Cl(A_n)\cap Cl(C_n))$, then the function $\operatorname{dist}(x,E)\sum_{\{n:\ c_n\leq x\}} d_n$ is continuous in x, so

$$\limsup_{t\to x} p(t)=\limsup_{t\to x} m(t)+\operatorname{dist}(x,E)\sum_{\{n:\ c_n\leq x\}} d_n \quad \text{and}$$

$$\liminf_{t\to x} p(t)=\liminf_{t\to x} m(t)+\operatorname{dist}(x,E)\sum_{\{n:\ c_n\leq x\}} d_n. \text{ Hence,}$$

$$S(p)-\bigcup_{n\in N} J_n-(Cl(A_n)\cap Cl(C_n))=A,$$

$$S^1(p)-\bigcup_{n\in N} J_n-(Cl(\tilde{A}_n)\cap Cl(C_n))=A^1,$$

$$C(p)-\bigcup_{n\in N} J_n-(Cl(A_n)\cap Cl(C_n))=B.$$

b/ If $x\in J_n\cap Cl(C_n)-Cl(A_n)$, then

$$\liminf_{t\to x} p(t) \leqslant \liminf_{t\to x}\left(m(t)+ \frac{1}{2}\operatorname{dist}(t,E)\right) = h(x) - \frac{1}{2}\operatorname{dist}(x,E) < h(x),$$

$h(x)= m(x) < p(x)$ and

$$\limsup_{t\to x} p(t) = \limsup_{t\to x}\left(m(t)+ \operatorname{dist}(t,E)\right)_{\{n:\ \sum c_n < t\}} d_n .$$

Since there exists a sequence (t_k) such that for each k $t_k > x$,

$t_k \in \left((J_n \cup B_n) - \mathrm{Cl}\ I_n\right) - C_n$, $\lim_{k\to\infty} t_k = x$, $\lim_{k\to\infty} h(t_k) = h(x)$ and

$\lim_{k\to\infty} \sum_{\{m:\ c_m < t_k\}} d_m > \sum_{\{m:\ c_m < x\}} d_m$, we have $\limsup_{t\to x} p(t) \geqslant \lim_{k\to\infty} p(t_k) > p(x)$

Therefore $\bigcup_{n\in N} J_n \cap \mathrm{Cl}(C_n) - \mathrm{Cl}(A_n) \subseteq R - (S(p) \cup S^1(p))$.

Similarly, $\bigcup_{n\in N} J_n \cap \mathrm{Cl}(A_n) - \mathrm{Cl}(C_n) \subseteq R - (S(p) \cup S^1(p))$.

Thus the function p has the following properties:

$$S(p) = A, \quad S^1(p) = A^1, \quad C(p) = B \quad \text{and} \quad T(p) = T^1(p) = \emptyset.$$

V. The final step consists in the construction of the function

$f: R \to R$, such that

$$S(f) = A, \quad S^1(f) = A^1, \quad C(f) = B, \quad T(f) = C \quad \text{and} \quad T^1(f) = C^1.$$

Because the set $C \cup C^1$ is countable, there exists a sequence $(e_n)_{n\in N}$ of all elements of $C \cup C^1$.

Let us define the function f as follows:

$$f(x)=\begin{cases} p(x)+\frac{1}{n} & \text{for} \quad x = e_n \text{ and } x \in C, \\ p(x)-\frac{1}{n} & \text{for} \quad x = e_n \text{ and } x \in C^1, \\ p(x) & \text{for} \quad x \in R-(C \cup C^1). \end{cases}$$

It is clear that f satisfies the conditions of the theorem.

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