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ON PROJECTIONS OF BIG PLANAR SETS

In [4] Steinhaus proved that the set of differences, A-B, contains an interval whenever A and B are linear sets of positive Lebesgue measure. The same conclusion holds when A and B have the property of Baire (Piccard [3]). Paraphrasing this result we see that the projection of $A \times B$ upon the line having equation $y = -x$ contains an interval. On the one hand, we strengthen this latter result to apply to various kinds of projections. On the other hand, we consider the general problem of whether or not a projection of an arbitrary planar set which is "big" in measure or category contains an interval. Our main result with respect to this second problem is that there exists a residual planar set no projection of which contains an interval. The analogous problem for sets of positive or full measure remains unsolved.

<u>Terminology</u>: If $f : R \to R$ and $E \subseteq R^2$ we say that the <u>f-projection of</u> E is the set $\{c : (f+c) \cap E \neq \emptyset\}$, that is the set of all c for which the graph of f+c intersects E. The f-<u>measure projection of</u> E is the set $\{c: \lambda_1(\text{dom}[(f+c) \cap E]) > 0\}$. Here λ_i denotes Lebesgue measure in R^i. The f-<u>category projection of</u> E is the set

$\{c: \mathrm{dom}[(f+c) \quad E]$ is of second category$\}$. We use the word *projection* to refer to any f-projection where f is linear.

When f is linear and E has positive measure (resp. is of second category) then, as is well known, the measure projection of E (resp. the category projection) fills up almost all of the projection of E in the sense of measure (resp. category). For Cartesian products this result as well as Steinhaus' theorem can be improved to the following

Theorem 1. Let A *and* B *be linear sets of positive measure. Suppose* f *maps sets of positive measure onto sets of positive measure. Then the* f*-measure projection of* $A \times B$ *contains an interval. Moreover, if* $A = B$ *and* f *is the identity function, then the interval contains* 0.

Proof: Let D denote the f-measure projection of $A \times B$. It suffices to show $D \cap C \neq \emptyset$ for any countable set C dense in R.

To this end fix a countable dense subset C of R. Suppose $D \cap C = \emptyset$. Then for each $c \in C$ $\lambda_1(\{x \in A : f(x) + c \in B\}) = 0$. Consequently the set $E = \{x \in A : f(x) + c \in B, c \in C\}$ also has measure 0. Choose P to be a closed subset of $A - E$ such that $\lambda_1(P) > 0$. According to Steinhaus' result the difference set $B - f(P)$ contains an interval. Hence, $C \cap (B - f(P)) \neq \emptyset$ which means that there exist $b \in B$, $c \in C$ and $p \in P \subseteq A$ such that $b = f(p) + c$.

Thus $p \in E$ which is a contradiction.

For the second part, assume that $A = B$ and $f(x) = x$ for all x. Assume that D does not contain an interval containing 0. Then there exists a sequence $\{c_n\}_{n=1}^{\infty}$ such that $c_n \to 0$ for which $\lambda_1(\{x \in A : x + c_n \in A\}) = 0$. Choose P to be a closed subset of $A - \text{dom } \bigcup_{n=1}^{\infty}((f+c_n) \cap (A \times A))$ with $\lambda_1(P) > 0$. According to [4] $P - P$ contains an interval centered at 0. But $P - P \subseteq D$, a contradiction.

Corollary 1. *If A and B are linear sets of positive measure, then each measure projection of $A \times B$, except possibly the vertical and horizontal projections, contains an interval.*

The exceptions above are necessary if one takes A and B to be nowhere dense sets of positive measure. Next we have the category analogue of Theorem 1.

Theorem 2. *Let A and B be linear sets having the property of Baire. Suppose f maps sets with the property of Baire onto sets with the property of Baire. Then the f-category projection of $A \times B$ contains an interval. Moreover, if $A = B$ and f is the identity function, then the interval contains 0.*

Proof: The proof is analogous to that of Theorem 1 using instead of Steinhaus' results the corresponding ones found in Piccard [3]. Here, the set E is of first category so that we can put $P = A - E$ which is a set with the property of Baire.

The category analogue of Corollary 1 can be improved to the following:

Theorem 3. *If A has the property of Baire and B is of second category, then any category projection of* $A \times B$, *except possibly the vertical and horizontal projections, contains an interval.*

Proof: Using the same proof as in Theorem 2 we need to show that $B - H$ contains an interval whenever H has the property of Baire. For this it suffices to show $(B-H) \cap C = \emptyset$. We have $H = (G \cup F_1) - F_2$ where G is open and F_1 and F_2 are of first category. Obviously there exists $c \in C$ such that $(G+c) \cap B$ is of second category. It follows that $c \in B - H$.

The conclusion of Theorem 3 is no longer valid if both A and B are just of second category as is shown by the following example. Moreover, it is not valid if B is taken to be an F_σ set c-dense in R (see Corollary 2 below).

Theorem 4. There exists a second category set A such that the projection of $A \times A$ onto any line with rational slope and rational intercept does not contain an interval.

Proof: Let Q denote the set of rational numbers and let $\{G_\alpha\}_{\alpha<c}$ be a well-ordering of all residual G_δ subsets of the line. Choose $a_0 \in G_0 - Q$. Suppose we have chosen $a_\alpha \in G_\alpha$ for all $\alpha < \beta$ in such a way that $a_\mu \notin Qa_\nu + Q$ whenever $\mu, \nu < \beta$. Choose $a_\beta \in G_\beta - \cup\{Qa_\alpha + Q : \alpha < \beta\}$. Put $A = \{a_\beta : \beta < c\}$. The set A is of second category because it intersects each residual G_δ set. Moreover, no line of the form $y = rx + s$ for $r, s \in Q$ can intersect $A \times A$. It follows that the projection of $A \times A$ onto any line with rational slope and rational intercept contains no rational number and hence no interval.

Note that the above result also shows that the category-projections fail to contain intervals for a dense set of directions.

It is unknown whether or not a second category set A can be found such that the (category) projection of $A \times A$ fails to have a non-empty interior in each direction.

The foregoing results suggest that arbitrary planar sets, not necessarily Cartesian products, will have projections with non-empty interiors provided the sets are "big" enough in measure or residual. The next theorem shows that this conjecture is invalid at least in the case that the set is residual.

<u>Theorem 5</u>: <u>There exists a</u> G_δ <u>residual planar set each projection of which has empty interior</u>.

<u>Proof</u>: For each n let O_n be an open dense subset of $[0,1]$ with $\lambda_1(O_n) = 3^{-n}$. Put $G_n = \{O_n + k : k$ an integer$\}$. Then clearly for any sequence $\{a_n\}_{n=1}^{\infty}$ in R

$$\lambda_1 \bigcup_{n=1}^{\infty}([0,1] \cap (G_n + a_n)) \leq \Sigma_{n=1}^{\infty}\lambda_1([0,1] \cap (G_n + a_n)) \leq \Sigma_{n=1}^{\infty}3^{-n} = \tfrac{1}{2}.$$

Put $P_n = R - G_n$. Then for any sequence $\{a_n\}_{n=1}^{\infty}$ in R the set $[0,1] \cap \bigcap_{n=1}^{\infty}(P_n + a_n)$ has measure $\geq \frac{1}{2}$ and therefore is non-empty.

Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the set of all rational numbers Q. For $x \in R$ define $F(x) = [0,1] \cap \bigcap_{n=1}^{\infty}(P_n - r_n x)$. Then for each $x \in R$ $F(x)$ is a non-empty compact set. Define $\pi_1 : R^2 \to R$ by $\pi_1(x,y) = x$. Then for any set K $\{x : F(x) \cap K \neq \emptyset\} = \pi_1(\bigcap_{n=1}^{\infty}\{(x,y) : y \in K \cap P_n - r_n x\})$. If K is compact this set is clearly compact. This means that the mapping F from R into the closed subsets of $[0,1]$ is upper-semi-continuous and according to Čoban [2] there exists a Borel 1 selection for F, i.e., there exists a Borel 1 function f such that $f(x) \in F(x)$ for all $x \in R$.

In particular, for all x, $f(x) \notin \cup_{n=1}^{\infty}(G_n - r_n x)$ so that $f(x) + r_n x \notin G_n$ for all n and x. Then the range of the function $f(x) + r_n x$ is nowhere dense in R for each fixed n.

For $m \in R$ and $r \in Q$ put $L_{m,r} = \{(x,y) : y = mx + f(m) + r\}$. Define $\pi(x,y,m) = (x,y)$. Then since f is a Borel 1 function $\{(x,y,m) : y = mx + f(m)\}$ is a G_δ set. Moreover, $\cup\{L_{m,o} : m \in R\} = \pi\{(x,y,m) : y = mx + f(m)\}$. Hence $\cup\{L_{m,0} : m \in R\}$ is the projection of a Borel set and therefore analytic. Since $L_{m,r} = L_{m,0} + (0,r)$ it follows that each $\cup\{L_{m,r} : m \in R\}$ is also analytic. Therefore, $A = R^2 - \cup\{L_{m,r} : m \in R,\ r \in Q\}$ is coanalytic and consequently measurable. Obviously A has the property that each projection has empty interior.

It remains to show that each $\cup\{L_{m,r} : m \in R\}$ is nowhere dense. Clearly we can take $r = 0$. Suppose, on the contrary, that it is dense somewhere, so that there exist intervals I and J such that $I \times J \subseteq \mathrm{cl} \cup\{L_{m,0} : m \in R\}$. For each n $(G_n + r_n) \cap J$ is a dense open subset of J. Pick an $r_n \in I - \{0\}$ and an open interval $S \subseteq (G_n + r_n) \cap J$. Then no $L_{m,0}$ can intersect the vertical segment $\{r_n\} \times S$. Otherwise $f(m) + mr_n \in S$ for some m, which contradicts the fact that $f(x) + r_n x \notin G_n$ for all x. Moreover, the set of intercepts of the lines $L_{m,0}$ for $m \in R$ must lie in the interval $\{0\} \times [0,1]$ since $0 \leq f(x) \leq 1$ for all x. Let T be the open triangle formed by the endpoints of $\{r_n\} \times S$ and the point of intersection of the two lines joining the upper (resp. lower) endpoints of $\{r_n\} \times S$ and $\{0\} \times [0,1]$. Clearly no $L_{m,0}$ can intersect T. On the other hand, T intersects $I \times J$. Hence, $\{L_{m,0} : m \in R\}$ can not be dense in $I \times J$, a contradiction.

Therefore A is a residual set having no projection with non-void interior. Hence, A contains a G_δ residual subset with the same property.

Corollary 2. *There exists a residual G_δ subset D of R and an F_σ subset E of R which is c-dense in R such that each projection of $D \times E$ has empty interior.*

Proof: Let A be the residual G_δ set of Theorem 5. From [1] we can find such sets D and E for which $D \times E \subseteq A$. Hence, no projection of $D \times E$ has nonempty interior.

It is unknown whether any set of full measure (or even positive measure) has a projection with non-empty interior. A set is of full measure if its complement has measure 0.

Note that the proof of Theorem 5 contains much more information than is necessary for the conclusion. The function need not be Borel measurable nor the set A measurable. However, we have included this information in the hope that it can be useful in ascertaining whether A has measure 0 or full measure.

We can, however, reduce the above question to the existence of a pathological function, as given in the next result.

Theorem 6. (1) If there exists a planar set of full measure each projection of which has empty interior, then

(a) There exists an $f : R \to R$ such that for almost all c $\lambda_1(\{f(x) + cx : x \in R\}) = 0$.

and

(b) There exists a set C of full measure in R and a function A from C into the null subsets of R such that for each x $\cap\{A(c) + cx : c \in C\} \neq \emptyset$.

(2) If there exists a Borel function $f : R \to R$ such that for almost all c $\lambda_1(\{f(x) + cx : x \in R\}) = 0$ then there exists a planar set of full measure each projection of which has empty interior.

A proof of this result can be constructed following the arguments in the proof of theorem 5.

One can also define projections at a point by taking the intersections of all lines joining this point and points in the set with a circle centered at the given point. Results similar to the above can be established by using the log and exp functions.

Bibliography

1. Ceder, J., On Globs, to appear in Acta Math. Acad. Sci. Hungar.

2. Čoban, M., Many-valued mappings and Borel sets II. Trans. Moscow Math. Soc. 22 (1970), 286-310.

3. Piccard S., Sur les ensembles de distance, Memoires Neuchâtel Université 1938-1939.

4. Steinhaus, H., Sur les distances des points des ensembles de mesure positive, Fund. Math. 1 (1920), 93-104.