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ASSOCIATED SETS AND CONTINUITY ROADS

Introduction. For a realvalued function of a real variable f, the associated sets of f are the sets $E^{\alpha}(f) = \{x|f(x) < \alpha\}$ and $E_{\alpha}(f) = \{x|f(x) > \alpha\}$ for real α. Many classes of functions can be characterized in terms of their associated sets. For example f is continuous if and only if all of its associated sets are open, and f is in the first Baire class (B_1) if and only if all of its associated sets are of type F_{σ}. A continuity road for a function f at a point x in its domain is a set E containing x and such that $f|_E$ is continuous at x. ($f|_E$ denotes the restriction of f to the domain E.) Many classes of functions are characterized by having continuity roads of a certain type at each point in their domains. For example, f is continuous if at each point it has a continuity road which is open, and f is approximately continuous if it has at each point a continuity road which is an F_{σ} set having the property that each point is a point of Lebesgue density of that set. We thus have two types of characterizations - in terms of associated sets and in terms of continuity roads. In Theorem 1 of this paper we obtain a result relating the two types of characterizations. In [4] Zahorski defined a nested sequence of classes of functions $\mathfrak{M}_i$, $i = 0,1,\ldots,5$. He then demonstrated that the $\mathfrak{M}_i$ classes are closely related to the class of ordinary derivatives Δ. Specifically (among many deep results in [4]) he established the inclusions $\mathfrak{M}_4 \supsetneq b\Delta \supsetneq b\mathfrak{M}_5$. ($b$ denotes bounded.) The $\mathfrak{M}_i$ classes are defined in terms of associated sets and $\mathfrak{M}_5$ was shown to equal the approximately continuous functions. Hence $\mathcal{M}_5$ admits both types of characterizations. We will apply Theorem 1 to the other $\mathcal{M}_i$ classes and will see that with the exception of $\mathcal{M}_4$, all the other $\mathcal{M}_i$, $i = 0,1,2,3$ also admit continuity road characterizations. In order to unify the study of the $\mathcal{M}_i$ and related classes, we introduce some terminology.

Definitions and Preliminaries. μ will denote Lebesgue measure, and all sets will be assumed Lebesgue measurable. $\mathcal{F}_\sigma$ will denote the class of all sets of type F_σ.

Definition. Let p be a property of sets (not necessarily all sets) with respect to sets containing them. If $A \subset B$, then we denote that A has (doesn't have) property p with respect to B by $A \subset_p B$ $(A \not\subset_p B)$. $\{x\} \subset_p B$ will be denoted by $x \in_p B$. p will be called a strong containment property if

i) $A \subset_p B \subset E$ implies $A \subset_p E$

ii) $A \subset B \subset_p E$ implies $A \subset_p E$

iii) If for each natural number n, $A_n \subset_p E_n$, then $\bigcup_{n=1}^{\infty} A_n \subset_p \bigcup_{n=1}^{\infty} E_n$.

Definition. If p is a strong containment property, we define the class of functions $\mathcal{M}_p$ by $f \in \mathcal{M}_p$ if and only if $E^\alpha(f) \subset_p E^\alpha(f) \in \mathcal{F}_\sigma$ and $E_\alpha(f) \subset_p E_\alpha(f) \in \mathcal{F}_\sigma$ for all α.

Definition. If p is a strong containment property and x is a real number, we define the class of functions $\mathcal{M}_p[x]$ by $f \in \mathcal{M}_p[x]$ if and only if there exists a set E such that $x \in E \subset_p E$ and $f|_E$ is continuous at x, i.e. E is a continuity road for f at x and $E \subset_p E$.

Now, if $A \subset B$, let $A \subset_{p_5} B$ mean that every point of A is a point of Lebesgue density of B. Then p_5 is easily seen to be a strong containment property and $\mathfrak{M}_{p_5} = \mathfrak{M}_5$. Also, $f \in \mathfrak{M}_{p_5}[x]$ means that f is approximately continuous at x and so we have $f \in \mathfrak{M}_5$ if and only if $f \in \mathfrak{M}_{p_5}[x]$ for all x. In order to investigate the analogous statements for the other $\mathfrak{M}_i$ classes, we formulate their definitions in terms of strong containment properties:

<u>Definitions</u>. If $A \subset B$, then $A \subset_{p_0(p_1,p_2)} B$ will mean that for $x \in A$ and $\varepsilon > 0$, the sets $(x-\varepsilon,x) \cap B$ and $(x,x+\varepsilon) \cap B$ are infinite (have cardinality c, have positive measure). It is clear that p_0, p_1 and p_2 are strong containment properties and $\mathfrak{M}_{p_0} = \mathfrak{M}_0$, $\mathfrak{M}_{p_1} = \mathfrak{M}_1$ and $\mathfrak{M}_{p_2} = \mathfrak{M}_2$. Zahorski showed that $\mathfrak{M}_0 = \mathfrak{M}_1 = \mathcal{D}\beta_1$ (Darboux-Baire 1 functions).

<u>Definition</u>. If $A \subset B$, then $A \subset_{p_3} B$ will mean that for $x \in A$ and each $k = 1,2,\ldots$, there exists $\varepsilon(x,k) > 0$ such that $\mu(I \cap B) > 0$ for all intervals I such that $\frac{\mu(I)}{d(x,I)} > \frac{1}{k}$ and $\mu(I) + d(x,I) < \varepsilon(x,k)$ ($d(x,I)$ denotes the distance from x to I). It is clear that p_3 is a strong containment property and $\mathfrak{M}_{p_3} = \mathfrak{M}_3$.

<u>Definition</u>. If $A \in F_\sigma$ and $A \subset B$, then $A \subset_{p_4} B$ will mean that there exists a sequence of closed sets A_n and a sequence of positive numbers η_n such that $A \subset \bigcup_{n=1}^{\infty} A_n$ and for each $x \in A \cap A_n$ and $k = 1,2,\ldots$ there exists $\varepsilon(x,k) > 0$ such that $\mu(I \cap B) > \eta_n \mu(I)$ for all intervals I for which $\frac{\mu(I)}{d(x,I)} > \frac{1}{k}$ and $\mu(I) + d(x,I) \leq \varepsilon(x,k)$. Clearly p_4 is a strong containment property and $\mathfrak{M}_{p_4} = \mathfrak{M}_4$.

All functions, unless otherwise specified, will be defined on $[0,1]$. In the sequel where properties are defined bilaterally, we make the necessary modifications at 0 and 1.

Main Results.

THEOREM 1. Let p be a strong containment property satisfying the three additional conditions:

a) if $E \subset_p E$ and G is open, then $E \cap G \subset_p E \cap G$;

b) p is defined pointwise, i.e., $A \subset_p B$ if and only if $x \in_p B$ for all $x \in A$;

c) suppose $\{E_n\}$ is a sequence of sets such that $E_{n+1} \subset E_n \subset_p E_n$ for all n and $\bigcap_{n=1}^{\infty} E_n = \{x\}$. Then there exist sequences of points $\{a_n\}$ increasing to x and $\{b_n\}$ decreasing to x such that $x \in_p \{x\} \cup \bigcup_{n=1}^{\infty} [(b_{n+1}, b_n) \cup (a_n, a_{n+1})] \cap E_n$.

Then, for a Baire 1 function f, the following are equivalent:

1) $f \in \mathfrak{M}_p[x]$ for all $x \in [0,1]$;

2) $f^{-1}(G) \subset_p f^{-1}(G)$ for all open sets G;

3) $g \circ f \in \mathfrak{M}_p$ for all continuous functions g.

PROOF: 1) implies 2). Let G be open, and let $x \in f^{-1}(G)$. Since $f \in \mathfrak{M}_p[x]$, there exists a set E such that $x \in E \subset_p E$ and $f|_E$ is continuous at x. Hence there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \cap E \subset f^{-1}(G)$. But by hypothesis a) we have $(x-\varepsilon, x+\varepsilon) \cap E \subset_p (x-\varepsilon, x+\varepsilon) \cap E \subset f^{-1}(G)$ and so $x \in_p f^{-1}(G)$. Hence since p is defined pointwise, we have $f^{-1}(G) \subset_p f^{-1}(G)$.

2) implies 1). Let $x \in [0,1]$ and we show $f \in \mathfrak{M}_p[x]$. Let $E_n = (x - \frac{1}{n}, x + \frac{1}{n}) \cap f^{-1}\{(f(x) - \frac{1}{n}, f(x) + \frac{1}{n})\}$. Then from a) and 2)

we see $E_{n+1} \subset E_n \subset_p E_n$ and $\bigcap_{n=1}^{\infty} E_n = \{x\}$. Hence by c) there exist sequences $\{a_n\}$ increasing to x and $\{b_n\}$ decreasing to x such that $x \in_p \{x\} \cup \bigcup_{n=1}^{\infty} [(b_{n+1},b_n) \cup (a_n,a_{n+1})] \cap E_n = E$. By iii) in the definition of strong containment property and a) above, we see that $E \subset_p E$ and evidently $f|_E$ is continuous at x.

2) implies 3). Let g be continuous and α be a real number. We need to show $E^{\alpha}(g \circ f) \subset_p E^{\alpha}(g \circ f) \in \mathcal{F}_{\sigma}$ and $E_{\alpha}(g \circ f) \subset_p E_{\alpha}(g \circ f) \in \mathcal{F}_{\sigma}$. But $E^{\alpha}(g \circ f) = \{x : g \circ f(x) < \alpha\} = f^{-1}(H)$, where $H = \{x : g(x) < \alpha\}$. Since g is continuous, H is open and so by hypothesis $f^{-1}(H) \subset_p f^{-1}(H)$. Since f is Baire 1, $f^{-1}(H) \in \mathcal{F}_{\sigma}$. Similarly, $E_{\alpha}(g \circ f) \subset_p E_{\alpha}(g \circ f) \in \mathcal{F}_{\sigma}$.

3) implies 2). Let $f \in \mathcal{M}_p$, G be open and we want to show $f^{-1}(G) \subset_p f^{-1}(G)$. We write $G = \bigcup_{n=1}^{\infty} (a_n,b_n)$ and then $f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(a_n,b_n)$. Now fix n and we show $f^{-1}(a_n,b_n) \subset_p f^{-1}(a_n,b_n)$. Let m be the midpoint of (a_n,b_n) and let $g(x) = \left(\frac{x - m}{a_n - m}\right)^2$. Then $g(x)$ is continuous and $(a_n,b_n) = \{x : g(x) < 1\}$. So $\{x : g \circ f(x) < 1\} = \{x : f(x) \in (a_n,b_n)\} = f^{-1}(a_n,b_n)$. But by hypothesis, $g \circ f \in \mathcal{M}_p$ and so $\{x : g \circ f(x) < 1\} \subset_p \{x : g \circ f(x) < 1\}$, i.e., $f^{-1}(a_n,b_n) \subset_p f^{-1}(a_n,b_n)$. Thus by condition iii) in the definition of a strong containment property, we have $f^{-1}(G) \subset_p f^{-1}(G)$. This completes the proof.

We observe that hypothesis c) was needed only in establishing 2) implies 1). Hence if p is any strong containment property satisfying a) and b) above, then by letting $g(x) = x$ in 3) of Theorem 1, we see that $f \in \mathcal{M}_p[x]$ for all x implies $f \in \mathcal{M}_p$.

We have thus established the following:

COROLLARY 1. If p is any of the properties p_0, p_1, p_2, p_3, and if f is any Baire 1 function such that $f \in \mathfrak{M}_p[x]$ for all $x \in [0,1]$, then $f \in \mathfrak{M}_p$.

It is worthwhile to note that many other classes of functions related to the derivatives can be defined in terms of strong containment properties. We give 2 examples:

Definition. If $A \subset B$, then $A \subset_{p_*} B$ will mean that for each $x \in A$, there exists $\eta_x > 0$ such that for each $k = 1,2,\ldots$ there exists $\varepsilon(x,k) > 0$ such that $\mu(I \cap B) > \eta_x \mu(I)$ for all intervals I for which $\mu(I) + d(x,I) \leq \varepsilon(x,k)$ and $\frac{\mu(I)}{d(x,I)} > \frac{1}{k}$.

Zahorski [4 pg. 52] asked if $\mathfrak{M}_{p*}$ (this author's notation) is the same as $\mathfrak{M}_4$. Lipiński [2] later answered in the negative showing $\mathfrak{M}_{p*} \subsetneq \mathfrak{M}_4$. However, it is evident that $p*$ satisfies hypotheses a) and b) of Theorem 1 and so for a Baire 1 function f, $f \in \mathfrak{M}_{p*}[x]$ for all $x \in [0,1]$ implies $f \in \mathfrak{M}_{p*}$.

Definition. If $A \subset B$, then $A \subset_{p_a} B$ will mean for each $x \in A$, $\lim_{n\to\infty} \frac{\mu(I_n)}{d(x,I_n)} = 0$ for all sequences of intervals $\{I_n\}$ for which $I_n \cap B = \phi$ for all n and $I_n \to x$. ($I_n \to x$ means every neighborhood of x contains all but finitely many of the intervals I_n.)

We remark that p_3 and p_a are similar to what has been called "non porosity" by some recent authors. See, for example, [1] for a definition.

It is easily seen that $\mathfrak{M}_3 \subset \mathfrak{M}_{p_a} \subset \mathfrak{M}_1$ and $\mathfrak{M}_{p_a} \cap \mathfrak{M}_2 = \mathfrak{M}_3$.

Again, p_a satisfies a) and b) of Theorem 1 and so for **Baire 1 functions f, $f \in \mathfrak{M}_{p_a}[x]$ for all $x \in [0,1]$ implies $f \in \mathfrak{M}_{p_a}$.**

We note that the property p_4 does not satisfy b) of Theorem 1. To see this, observe that $x \in_{p_4} B$ if and only if $x \in_{p_*} B$. Then since p_* satisfies b), p_4 cannot for otherwise, p_4 and p_* would be equivalent. It is still undetermined whether or not $f \in \mathfrak{M}_{p_4}[x]$ for all $x \in [0,1]$ implies $f \in \mathfrak{M}_{p_4}$.

We now investigate the converse of Corollary 1.

COROLLARY 2. If $f \in \mathfrak{M}_{p_1}$, then $f \in \mathfrak{M}_{p_1}[x]$ for all $x \in [0,1]$.

PROOF: p_1 evidently satisfies the hypotheses of Theorem 1. Since $\mathfrak{M}_{p_1} = \mathcal{D}\beta_1$ and this class is closed under composition on the left by continuous functions, the result follows from Theorem 1.

COROLLARY 3. If $f \in \mathfrak{M}_{p_0}$, then $f \in \mathfrak{M}_{p_0}[x]$ for all $x \in [0,1]$.

PROOF: $\mathfrak{M}_{p_0} = \mathfrak{M}_{p_1}$ (see [4]) and clearly $\mathfrak{M}_{p_1}[x] \subset \mathfrak{M}_{p_0}[x]$, so the result follows from Corollary 2.

THEOREM 2. If $f \in \mathfrak{M}_{p_2}$, then $f \in \mathfrak{M}_{p_2}[x]$ for all $x \in [0,1]$.

PROOF: p_2 evidently satisfies the hypotheses of Theorem 1. Since $\mathfrak{M}_{p_2} = \mathfrak{M}_2$, and any $f \in \mathfrak{M}_2$ is known to possess the Denjoy-Clarkson property (see [3]), we have $f^{-1}(G) \subset_{p_2} f^{-1}(G)$ for all open sets G. The result follows from Theorem 1.

THEOREM 3. If $f \in \mathfrak{M}_{p_3}$, then $f \in \mathfrak{M}_{p_3}[x]$ for all $x \in [0,1]$.

PROOF: p_3 evidently satisfies a) and b) of Theorem 1. We wish to use Theorem 1 and so must verify that p_3 satisfies c). So suppose $\{E_n\}$ is a sequence of sets, $E_{n+1} \subset E_n \subset_{p_3} E_n$ and $\bigcap_{n=1}^{\infty} E_n = \{x\}$. We construct the sequence $\{b_n\}$, the construction of $\{a_n\}$ being analogous. We choose b_n by induction to satisfy:

1) $x < b_n < x + \frac{1}{n}$

2) $b_n < b_{n-1}$

3) $b_n \in E_{n+1}$

4) $\mu(I \cap E_n) > 0$ for all I for which $\frac{\mu(I)}{d(x,I)} > \frac{1}{n}$ and $\mu(I) + d(x,I) < b_n - x$.

Since $x \in E_2 \subset E_1 \subset_{p_3} E_1$, there exists $b_1 \in E_2$ such that $x < b_1 < x + \frac{1}{2}$, and $\mu(I \cap E_1) > 0$ for all I for which $\frac{\mu(I)}{d(x,I)} > 1$ and $\mu(I) + d(x,I) < b_1 - x$. Assume $b_1, b_2, \ldots, b_k$ have been chosen to satisfy 1), 2), 3), and 4) above. Since $x \in E_{k+2} \subset E_{k+1} \subset_{p_3} E_{k+1}$, there exists $b_{k+1} \in E_{k+2}$ such that $x < b_{k+1} < x + \frac{1}{k+1}$, $b_{k+1} < b_k$ and $\mu(I \cap E_{k+1}) > 0$ for all I for which $\frac{\mu(I)}{d(x,I)} > \frac{1}{k+1}$ and $\mu(I) + d(x,I) < b_{k+1} - x$. This completes the induction.

Now let $F_n = (b_{n+1}, b_n) \cap E_n$ and $F = \bigcup_{n=1}^{\infty} F_n$. Then evidently $F \subset_{p_3} F$ and we need to show that $x \in_{p_3} F \cup \{x\}$. Given k, we choose $\varepsilon < b_k - x$ and then $\frac{\mu(I)}{d(x,I)} > \frac{1}{k}$ and $\mu(I) + d(x,I) < \varepsilon$ imply $\mu(I) + d(x,I) < b_k - x$. Hence either $I \subset (b_{n+1}, b_n)$ or I contains b_n for some $n > k$. In either case $\mu(I \cap F_n) > 0$ or $\mu(I \cap F_{n-1}) > 0$ and so $\mu(I \cap F) > 0$. Thus p_3 satisfies hypothesis c) in Theorem 1.

Suppose $f \in \mathfrak{M}_{p_3}$ but $f \notin \mathfrak{M}_{p_3}[x]$ for some $x \in [0,1]$. By using Theorem 1 and the fact that p_3 is defined pointwise, there exists an interval (a,b) and a point $x_0 \in f^{-1}(a,b)$ such that $x_0 \notin_{p_3} f^{-1}(a,b)$. Thus there exists a sequence of intervals $\{I_n\}$ such that $x_0 \notin I_n$, $I_n \to x_0$,

$\frac{\mu(I_n)}{d(x_0,I_n)} > n > 0$ for all n and $\mu(I_n \cap f^{-1}(a,b)) = 0$ for all n.

Now, for any n, $f|_{I_n} \in \mathcal{M}_{p_2}$. Hence, by Theorem 2, $f|_{I_n} \in \mathcal{M}_{p_2}[z]$ for all $z \in I_n$ and so $I_n \cap f^{-1}(a,b) \subset_{p_2} I_n \cap f^{-1}(a,b)$ for all n. But $\mu(I_n \cap f^{-1}(a,b)) = 0$ for all n, and so $I_n \cap f^{-1}(a,b) = \emptyset$ for all n. Since f is Darboux, we must have that for each n, either $f(z) \geq b$ for all $z \in I_n$ or $f(z) \leq a$ for all $z \in I_n$. Hence, there exists a subsequence $\{I_{n_k}\}$ of $\{I_n\}$ such that $I_{n_k} \to x_0$ and $f(z) \geq b$ for all $z \in I_{n_k}$ and all k (or $f(z) \leq a$ for all $z \in I_{n_k}$ and all k). Thus either $x_0 \notin_{p_3} \{x : f(x) < b\}$ or $x_0 \notin_{p_3} \{x : f(x) > a\}$ contradicting $f \in \mathcal{M}_{p_3}$. This proves the theorem.

It is worth noting that by combining Theorems 2 and 3 with Theorem 1, we obtain a proof of the fact that $\mathcal{M}_2$ and $\mathcal{M}_3$ are closed under outside composition with continuous functions.

We now construct a function f defined on $(-1,1)$ such that $f \in \mathcal{M}_{p_4}$ but $f \notin \mathcal{M}_{p_4}[0]$. For each $n = 0,1,2,\ldots$, let $\frac{1}{2^{n+1}} = x_n^0 < x_n^1 < x_n^2 < \ldots < x_n^{4^n} = \frac{1}{2^n}$ be a partition of $\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$ into congruent subintervals. Now let (a_n^i, b_n^i), $i = 1,2,\ldots,4^n$, be a set of non-overlapping intervals of equal lengths such that:

1) the midpoint of $(a_n^i, b_n^i) = x_n^i$;

2) $\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{4^n} (a_n^i, b_n^i)$ has zero density at the origin.

For notational convenience, we say $b_{n+1}^{4^{n+1}} = b_n^0$.

Now define f on $(0,1)$ by:

$$f = \begin{cases} 1 \text{ on } [b_n^i, a_n^{i+1}] & \text{if } i \text{ is odd;} \\ -1 \text{ on } [b_n^i, a_n^{i+1}] & \text{if } i \text{ is even;} \\ \text{extended linearly between 1 and } -1 \text{ on } (a_n^i, b_n^i) \\ \quad \text{so as to make } f \text{ continuous on } (0,1). \end{cases}$$

Let $f(x) = f(-x)$ for $x \in (-1,0)$, and let $f(0) = 0$. We first show that f is the derivative of its integral. For $x \neq 0$, $f(x)$ is continuous and so it remains to show $\lim\limits_{x\to 0} \frac{1}{|x|} \int_0^x f(t)dt = 0$. Now, if $|x| = \frac{1}{2^n}$, it is clear by the construction of f that $\int_0^x f(t)dt = 0$ so suppose $|x| \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$. Then by the symmetry of f, we have

$$\left|\int_0^x f(t)dt\right| = \left|\int_0^{\frac{1}{2^{n+1}}} f(t)dt + \int_{\frac{1}{2^{n+1}}}^x f(t)dt\right| = \left|\int_{\frac{1}{2^{n+1}}}^x f(t)dt\right|.$$ And

$\left|\int_{\frac{1}{2^{n+1}}}^x f(t)dt\right| < \frac{1}{4^n}$ by the construction of f. Now, since

$|x| > \frac{1}{2^{n+1}}$, we have $\frac{1}{|x|} < 2^{n+1}$ and so $\frac{1}{|x|} \left|\int_{\frac{1}{2^{n+1}}}^x f(t)dt\right|$

$< \frac{2^{n+1}}{4^n} = \frac{1}{2^{n-1}}$. Now, as $x \to 0$, $n \to \infty$ and so we have $\lim\limits_{x\to 0} \frac{1}{|x|} \int_0^x f(t)dt = 0$.

So f is bounded and everywhere the derivative of its integral, and since bounded derivatives are all in the class $\mathfrak{M}_4$ we have

$f \in \mathcal{M}_4$. But it is easy to see $f \notin \mathcal{M}_{p_4}[0]$. Let $A = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{4^n} (a_n^i, b_n^i)$ and $-A = \{x: -x \in A\}$. We observe that $0 \in f^{-1}(-1,1) = A \cup -A$ which has zero density at 0 and hence there is no set E such that $0 \in E \subset_{p_4} E \subset f^{-1}(-1,1)$. Thus there is no set E such that $0 \in E \subset_{p_4} E$ for which $f|_E$ is continuous at 0.

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