J. G. Ceder

Department of Mathematics
University of California, Santa Barbara, CA 93106

T. L. Pearson[1]

Department of Mathematics
Acadia University, Wolfville, N.S. CANADA BOP 1XO

A SURVEY OF DARBOUX BAIRE 1 FUNCTIONS

Introduction: The purpose of this article is to summarize the important results and open problems concerning Darboux Baire 1 functions from a real interval into the reals. The class of such Darboux Baire 1 functions, denoted by $\mathcal{D}\mathcal{B}_1$, contains subclasses of functions important for differentiation theory, among which are the derivatives, Δ, and the approximately continuous functions, $\mathcal{A}$. An extensive treatment of these is found in [3] and consequently we will mention only those results on Δ and $\mathcal{A}$ which are relevant to our treatment of $\mathcal{D}\mathcal{B}_1$ functions. Nor will we survey the facts on the much larger and considerably less structured class of all Darboux functions $\mathcal{D}$ or the classes $\mathcal{D}\mathcal{B}_\alpha$ where $\alpha > 1$. A survey of these can be found in [4] or partially in [3]. However, we do cover the hitherto neglected subclasses of upper semi-continuous and lower semi-continuous Darboux functions, $\mathcal{D}$ usc and $\mathcal{D}$ lsc.

---

[1] The second author was supported in part by the Natural Sciences and Engineering Research Council of Canada.

Notation and Terminology: In the sequel we will assume that all functions are defined on a non-degenerate real interval, denoted by I, unless otherwise specified.

The right and left cluster sets of f at x are denoted by $K^+(f,x)$ and $K^-(f,x)$. Moreover, $K(f,x) = K^+(f,x) \cup K^-(f,x)$. The sets of continuity points and approximate continuity points of f are denoted by C(f) and A(f). If $\mathcal{E}$ and $\mathcal{H}$ are classes of functions, then $\mathcal{E} + \mathcal{H}$ and $\mathcal{E} \cdot \mathcal{H}$ denote all f + g (resp. f·g) where $f \in \mathcal{E}$ and $g \in \mathcal{H}$. And lim is the set of all pointwise limits of sequences in $\mathcal{E}$. The class of all bounded f in $\mathcal{E}$ is denoted by $b\mathcal{E}$. Lebesgue measure is denoted by λ and the class of all continuous functions is denoted by $\mathcal{C}$. A set A is null if $\lambda(A) = 0$. The graph of f is denoted by gr f. If for all $\varepsilon > 0$ the set

$$\{z : |f(z) - L| < \varepsilon\}$$

has density 1 at x, then we write $\lim_{z \to x} \text{ap } f(z) = L$.

We say that a function f has a p-system with respect to c-dense (resp. dense, density) containment if there exists a system of perfect sets $\{A_n^{y_r}, A_{y_r}^n\}$ where $n \geq r$ and n and r are positive integers and $\{y_r\}_{r=1}^{\infty}$ is an enumeration of the rational numbers satisfying the following:

(1) $\bigcup_{n=r}^{\infty} A_n^{y_r} = [f > y_r]$ (1)' $\bigcup_{n=r}^{\infty} A_{y_r}^n = [f < y_r]$

(2) $A_n^{y_r} \subseteq_p A_{n+1}^{y_r}$ (2)' $A_{y_r}^n \subseteq_p A_{y_r}^{n+1}$

and if $y_s < y_t$ and $n \geq \max\{s,t\}$ then

(3) $A_n^{y_s} \subseteq_p A_n^{y_t}$ (3)' $A_{y_t}^n \subseteq_p A_{y_s}^n$

where $B \subseteq_p C$ means each point of B is a bilateral c-limit (resp. limit, density point) of C. In this case we say that B is c-dense (resp. dense, density) contained in C.

A collection of non-empty closed sets $\{A(\alpha) : \alpha \in J\}$, where J is an interval, is called a hierarchy with respect to c-dense (resp. dense, density) containment if $\alpha < \beta$ implies A_α is c-dense (resp. dense, density) contained in A_β.

Since f is usc if and only if $-f$ is lsc all the results in the sequel on usc functions yield corresponding results on lsc functions whose formulations we will omit.

In the interest of brevity we will not give references to old established results on Darboux functions, ones which have been documented in [4] or [3].

We begin by citing a theorem which is very useful for constructing functions in $\mathcal{D}\mathcal{B}_1$ or in $\mathcal{A}$.

Theorem A [1]. *If $\{A(\alpha) : \alpha \in [1,\infty)\}$ is a hierarchy of closed sets with respect to c-dense (resp. density) containment, and if f is defined as*

$$f(x) = \begin{cases} (\inf\{\alpha : x \in A(\alpha)\})^{-1} & \text{if } x \in \bigcup_{\alpha \geq 1} A(\alpha) \\ 0 & \text{if } x \notin \bigcup_{\alpha \geq 1} A(\alpha), \end{cases}$$

then $f \in \mathcal{D}$ usc (resp. $\mathcal{A}$ usc) and $[f > 0] = \bigcup_{\alpha \geq 1} A(\alpha)$.

Within the class of Baire 1 functions, $\mathcal{B}_1$, there is a variety of useful characterizations of being Darboux. The following theorem contains some of these. Others may be found in [3] or [4].

Theorem B. *Suppose* f *is Baire* 1. *Then each of the following conditions is necessary and sufficient for* f *to be Darboux*:

(b_1) f *is the uniform limit of a sequence of Darboux functions*;

(b_2) *for all* x, $K^+(f,x)$ *and* $K^-(f,x)$ *are intersecting closed intervals*;

(b_3) *for all* x, $f(x) \in K^+(f,x)$

(b_4) gr f *is connected*;

(b_5) gr f *is bilaterally* c-*dense (resp. dense) in itself*;

(b_6) *each neighborhood of* gr f *contains the graph of a continuous function* [9];

(b_7) *for each* x *there exists a perfect set* P *having* x *as a bilateral limit point such that* $f|P$ *is continuous at* x;

(b_8) *for each* α *each of the sets* $[f > \alpha]$ *and* $[f < \alpha]$ is bi*laterally* c-*dense* (*resp. dense*) *in itself*;

It is also possible to characterize Darboux Baire 1 functions as shown by

Theorem C. Each of the following is a necessary and suffi*cient condition for* f *to be a Darboux Baire 1 function*:

(c_1) *there exists a homeomorphism* h *from* I *onto* I *such that* $f \circ h \in \Delta$ (resp. $\mathcal{A}$) [24] (*or, equivalently*, f *is the derivative of its integral with respect to some nonatomic Lebesgue-Stieltjes measure* [3, p.42]);

(c_2) for any α each of the sets $[f > \alpha]$ and $[f < \alpha]$ is a bilaterally c-dense (resp. dense) in itself F_σ;

(c_3) for each closed subinterval J there exists $W(J) \in \text{int } J$ such that if $\{J_n\}_{n=1}^\infty$ is any sequence of such subintervals, then whenever $x \in J_k$ for all K and $\text{diam } J_k \to 0$ then $\lim_{k \to \infty} f(W(J_k)) = f(x)$;

(c_4) there exists a p-system for f with respect to c-dense (resp. dense) containment [2];

(c_5) there exist sequences of $\mathcal{DB}_1$ functions $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ such that $f_n \uparrow f$ and $g_n \downarrow f$.

There is also a characterization of $\mathcal{DB}_1$ in terms of the structure of the cluster sets [29], which we omit because of its complexity. Characterization (c_5) is new but its proof is immediate.

Some useful characterizations of usc are given by

Theorem D. Each of the following is a necessary and sufficient condition that f be upper semi-continuous and Darboux:

(d_1) for each x, $f(x) = \overline{\lim}_{z \to x^+} f(z) = \overline{\lim}_{z \to x^-} f(z)$;

(d_2) there exists a homeomorphism h from I onto I such that $f \circ h \in \Delta\text{usc}$ (resp. $\mathcal{A}\text{usc}$);

(d_3) for each α, $[f < \alpha]$ is open and $[f > \alpha]$ is a bilaterally c-dense (resp. dense) in itself F_σ set;

(d_4) $\{[f \geq \alpha] : \alpha \in f(I)\}$ is a hierarchy of closed sets with respect to c-dense containment [1].

Characterizations (d_2) and (d_3) are trivial consequences of (c_1)and (c_2) respectively. Moreover, (d_4) is immediate and may be considered a simplified version of (c_4) applied to the usc case.

The class of approximately continuous functions has characterizations similar to some of those for $\mathcal{D}\mathcal{B}_1$ by replacing the notion of c-limit point by that of a point of density.

Theorem E. Each of the following is a necessary and sufficient condition for f to be approximately continuous:

(e_1) f is continuous relative to the density topology (i.e. the inverse image of an open set is density-contained in itself);

(e_2) for all x, $f(x) = \lim_{z \to x} \text{ap } f(z)$;

(e_3) there exists a p-system for f with respect to density-containment [2].

Theorem F. Each of the following is a necessary condition for f to be upper semi-continuous and approximately continuous:

(f_1) $\{[f \geq \alpha] : \alpha \in f(I)\}$ is a hierarchy of closed sets with respect to density-containment;

(f_2) for all x, $f(x) = \lim_{z \to x} \text{ap } f(z) = \overline{\lim_{z \to x}} f(z)$.

In contrast to $\mathcal{A}$ the existence of characterizations for Δ is an open question with the exception of a characterization in terms of interval functions (see [20]) consisting of an augmentation of (c_3). However, when restricted to upper semi-continuity we have $b\Delta usc = b\mathcal{A}usc$.

One general problem is to find additional "simple", "non-contrived" characterizations for $\mathcal{D}\mathcal{B}_1$. However, there does not seem to exist a characterization in terms of the topological nature of the graph (e.g., a function with a connected G_δ graph need not be in $\mathcal{D}\mathcal{B}_1$ [18]). Nor does there seem to exist a simple "local" characterization of $\mathcal{D}\mathcal{B}_1$, even though Darbouxness does have a local characterization (see [4]). In [32] Baire 1 functions are characterized as having graphs being the intersection of a sequence of simply connected open sets. Does there exist a characterization of $\mathcal{D}\mathcal{B}_1$ having the graph as the intersection of a sequence of "special" simply connected open sets?

Another problem: Can the characterizations in terms of p-systems and hierarchies be exploited to yield simpler proofs of known results?

The class $\mathcal{D}\mathcal{B}_1$ as well as each of the subclasses $\mathcal{D}$usc, $\mathcal{A}$, and Δ are closed under taking uniform limits, but not under taking pointwise limits as shown by

Theorem G [6, 24, 25]. $\mathcal{B}_2 = \lim b\mathcal{D}\mathcal{B}_1 = \lim b \quad = \lim b\Delta$.

An open question is: What is $\lim \mathcal{D}$usc? Whereas $\mathcal{A}$ and Δ are closed under taking sums, $\mathcal{D}\mathcal{B}_1$ and $\mathcal{D}$usc are not. However, we do have

Theorem H [6]. $\mathcal{B}_1 = \mathcal{D}\mathcal{B}_1 + \mathcal{D}\mathcal{S}_1$.

This result suggests several questions: <u>What are</u> $\mathcal{D}$ usc + $\mathcal{D}$ usc <u>and</u> $\mathcal{D}$ usc + $\mathcal{D}$ lsc? In fact, more generally, <u>is</u> $\mathcal{B}_1$ = usc + lsc?

Other facts worth mentioning are: $\mathcal{C} + \mathcal{D}\mathcal{B}_1 = \mathcal{D}\mathcal{B}_1 = \mathcal{C} \cdot \mathcal{D}\mathcal{B}_1$.

As far as products are concerned $\mathcal{Q}$ is closed under products but Δ is not. There is an interesting representation for $\mathcal{B}_1$, namely, $\mathcal{B}_1 = \Delta + \Delta \cdot \Delta$ [27].

Another question: <u>is it true that</u> $f \in \mathcal{D}\mathcal{B}_1 \cdot \mathcal{D}\mathcal{B}_1$ <u>if and only if</u> $f \in \mathcal{B}_1$ <u>and</u> f <u>takes on the value</u> 0 <u>in each interval in which it changes sign?</u> The analogous result is true for $\mathcal{D}$ [12]. Also, <u>what is</u> $\mathcal{D}$ usc$\cdot$ $\mathcal{D}$ usc?

The following theorem, used to prove Theorem H in [6], shows that each Baire 1 function can be modified on a "small" set so as to become Darboux Baire 1.

<u>Theorem</u> I [6]. <u>Suppose</u> $f \in \mathcal{B}_1$ <u>and</u> E <u>is of first category. Then there exists</u> $g \in \mathcal{D}\mathcal{B}_1$ <u>such that</u> $[f \neq g]$ <u>is null and of first category and disjoint from</u> E <u>and such that</u> $f+g \in \mathcal{D}\mathcal{B}_1$.

<u>Is the analogous statement true for</u> $\mathcal{D}$ usc? (See [19]). <u>Moreover, given a sequence</u> $\{f_n\}_{n=1}^{\infty}$ <u>in</u> $\mathcal{B}_1$, <u>does there exist a</u> $g \in \mathcal{D}\mathcal{B}_1$ <u>such that</u> $f_n + g \in \mathcal{D}\mathcal{B}_1$ <u>for all</u> n?

The next result characterizes those Baire 1 functions which can be modified on a countable set to give a Darboux Baire 1 function.

Theorem J [8]. Suppose $f \in \mathcal{B}_1$. Then, there exists $g \in \mathcal{DB}_1$ such that $[f \neq g]$ is countable if and only if

(1) for each x, $K(f,x)$ is a non-void closed interval, and

(2) for each $\varepsilon > 0$ the set $\{x : \mathrm{dist}(K(f,x), f(x)) \geq \varepsilon\}$ is a countable G_δ set.

Despite the non-additivity of $\mathcal{DB}_1$ we have:

Theorem K [7]. Suppose $f,g \in \mathcal{DB}_1$ and $f < g$. Then there exists $h \in \mathcal{DB}_1$ such that $f < h < g$.

The analogous statement is also true for $\mathcal{D}$usc [19]. An open question is: What are necessary and sufficient conditions on arbitrary f and g, where $f < g$, in order to insure that a $\mathcal{DB}_1$ function can be inserted between them? (see [11]).

A somewhat related question is: Is a given $h \in b\mathcal{B}_1$ the average of some $f,g \in \mathcal{DB}_1$ such that $f < g$? This is not true without the boundedness restriction [8].

It is easy to see that the maximum of two $\mathcal{DB}_1$ functions need not be Darboux. What is the class $\max\{\mathcal{DB}_1, \mathcal{DB}_1\}$? A characterization of $\max\{\mathcal{D}, \mathcal{D}\}$ was given in [8]. Moreover, $\max\{\mathcal{D}\text{usc}, \mathcal{DB}_1\} = \mathcal{DB}_1$ and $\max\{\mathcal{D}\text{usc}, \mathcal{D}\text{usc}\} = \mathcal{D}\text{usc}$ [16].

Another, as yet unapplied, result is

Theorem L [13]. Let $f \in \mathcal{DB}_1$ from I onto J. Then there exists $g \in \mathcal{DB}_1$ from J onto I such that $f(g(y)) = y$ for all $y \in J$.

If we restrict ourselves to bounded functions, then $b\mathcal{DB}_1$ with the sup norm becomes a non-separable [14] complete metric space. Moreover,

$$\mathcal{C} \subseteq b \quad \subseteq b\Delta \subseteq b\mathcal{DB}_1,$$
$$\subseteq b\mathcal{D}usc \subseteq b\mathcal{DB}_1,$$

and in each inclusion the smaller class is a nowhere dense closed subspace larger. Let us say a property ϕ is typical in $b\mathcal{DB}_1$, if the class of all functions satisfying ϕ is residual in $b\mathcal{DB}_1$. Using this terminology, the properties of C, $b\mathcal{A}$, $b\Delta$, and $\mathcal{D}usc$ all fail to be typical.

So, a natural question to pose is: What kind of properties of $b\mathcal{DB}_1$ functions are typical?

The next theorem lists some known typical properties.

Theorem M. Each of the following properties is typical in $b\mathcal{DB}_1$:

(m_1)	$\lambda(f(A(f))) = 0$	[15]
(m_2)	$\lambda(\mathrm{cl}\ f(C(f))) = 0$	[15]
(m_3)	$\lambda(\mathrm{cl}\ f(A(f))) = 0$	[15]
(m_4)	$\lambda(C(f)) = 0$	[23]
(m_5)	$\mathrm{card}\ f(C(f)) = 2^{\aleph_0}$	[23]

(m_6) $I - C(f)$ is dense in I

(m_7) $f^{-1}(y)$ is null and nowhere dense for all y [23]

(m_8) the function $f(x) + rx$ is nowhere monotonic for each r

(m_9) f has both $+\infty$ and $-\infty$ as derived numbers at each point [14]

(m_{10}) f has an infinite derived number on both the right and left at each point [14]

(m_{11}) each real number is a derived number at each point [30]

(m_{12}) there is a residual set $E \subseteq I$ such that for each $x \in E$ each real number is a derived number on both the left and right at x [30]

(m_{13}) there exists a residual set $E \subseteq I$ such that the intersection of the line $y = mx + b$ with gr f is a dense in itself boundary set whenever m is rational and $b \in E$.

(m_{14}) the set of all (m,b) for which $y = mx + b$ fails to intersect gr f in a dense in itself set is null and of first category. [31]

Characterizations (m_6), (m_8), and (m_{13}) are new. The proof of (m_6) is straightforward and the other two follow immediately from (m_6) and results in [17]. Note, relative to (m_{13}) and (m_{14}), that the empty set is dense in itself.

There are a number of attractive candidates for a typical property in $b\mathcal{D}\mathcal{B}_1$. One of these, suggested in [23], is:

(n_1) for all y, cl $f^{-1}(y)$ is nowhere dense (and null).

Other related candidates are the following:

(n_2) *each number is a derived number on both the right and left at each point*;

(n_3) f *has no local unilateral extrema*;

(n_4) f *is open* (i.e., *has no local extrema*);

(n_5) *for each* y, $f^{-1}(y)$ *is bilaterally dense in itself*;

(n_6) *zero is a bilateral derived number at each point*;

Clearly, the following implications hold:

$$n_2 \longrightarrow n_3 \longrightarrow n_4, \qquad n_3 \longrightarrow n_5 \longrightarrow n_6$$

In [5] it is shown that the class of f satisfying (n_2) [and hence, (n_i), $i \geq 3$] is dense in $b\mathcal{DB}_1$ and therefore is likely to be typical.

Post-script: M. Chlebík has announced that the following property is typical for a $b\mathcal{DB}_1$ function f: f *attains its maximum on each compact subinterval at precisely one point.* It is easy to see that this implies that (n_2) through (n_5) are all non-typical.

Another general problem is to find typical functions in $busc\mathcal{D}$ and $b\mathcal{A}$. For some results on this problem see [9] [23] [19] [30] and [31].

In conclusion, we mention another interesting subclass of $\mathcal{DB}_1$, namely, $\mathcal{DB}_1^*$. We say that $f \in \mathcal{B}_1^*$ if for each closed set $P \subseteq I$ there exists an open interval (a,b) such that f is continuous on $(a,b) \cap P$. Obviously, $\mathcal{B}_1^* \subseteq B_1$. It is easy to see that $f \in \mathcal{DB}_1^*$ if and only if there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of closed subsets of I such that $I = \bigcup_{n=1}^{\infty} K_n$ and $f|K_n$ is continuous for each n.

Theorem M [21]. If $f \in \mathcal{DB}_1^*$, then f has a local extremum and $f(I) - f(C(f))$ is countable and nowhere dense.

The class $\mathcal{DB}_1^*$ is not closed in $\mathcal{DB}_1$ (e.g., consider the sequence $\{f_n\}_{n=1}^{\infty}$, where $f_n(x) = \Sigma_{k=1}^{n} 2^{-k}\sin(x-r_k)^{-1}$ for $x \neq r_k$ and $f_n(r_k) = 0$ on $(0,1)$, where $\{r_k\}_{k=1}^{\infty}$ is an enumeration of the rationals in $(0,1)$.) Moreover, $\mathcal{DB}_1^*$ is nowhere dense in $\mathcal{DB}_1$; this can be shown using a construction employing sequences similar to that given above.

It would be interesting to find characterizations of $\mathcal{DB}_1^*$ functions. It is known, however, that $\mathcal{DB}_1^*$ functions cannot be characterized by associated sets [28].

The $\mathcal{DB}_1^*$ analogue of Theorem J is true [22]. In general, however, little seems to be known about $\mathcal{DB}_1^*$ functions so that all the questions posed above concerning $\mathcal{DB}_1$ and $\mathcal{D}$usc functions can also be formulated for $\mathcal{DB}_1^*$ functions.

BIBLIOGRAPHY

[1] Agronsky, S.J., Ph.D. Thesis, Characterizations of Certain Subclasses of the Baire Class 1,. University of California, Santa Barbara, 1975.

[2] __________, A generalization of a theorem of Maximoff and applications, Trans. Amer. Math. Soc. 273 (1982), 767-779.

[3] Bruckner, A.M., Differentiation of Real Functions, Springer-Verlag, Berlin-New York, 1978. MR 80h : 26002.

[4] Bruckner, A.M.; Ceder, J.G., Darboux continuity, Jber. Deutsch. Math.-Verein 67 (1964/65), 93-117. MR32 #4217.

[5] __________, On a special class of Darboux Baire-one functions, to appear in Periodica Math. Hung. 14 (1983).

[6] Bruckner, A.M.; Ceder, J.G.; Keston, R., Representations and approximations by Darboux functions of the first class of Baire, Rev. Roumaine Math. Pures Appl. 13 (1968), 1246-1254. MR39 #4333.

[7] Bruckner, A.M.; Ceder, J.G.; Pearson, T.L., On the insertion of Darboux, Baire-one functions, Fund. Math. 80 (1973), 157-167. MR48 #6329.

[8] __________, On Darboux Functions. Rev. Roumaine Math. Pures Appl. 19 (1974), 977-988. MR50 #4846.

[9] Bruckner, A.M., Petruska Gy., Some typical results on bounded Baire 1 functions to appear in Acta Math Acad Sci. Hung.

[10] Brown, Jack B., Almost continuous Darboux functions and Reed's pointwise convergence criteria, Fund. Math. 86 (1974), 1-7. MR50 #4845.

[11] Ceder, J.G., Characterizations of Darboux selections, Rend. Circ. Mat. Palermo 30 (1981), 461-470.

[12] __________, On factoring a function into a product of Darboux functions, Rend. Circ. Mat. Palermo 31 (1982), 16-22.

[13] Ceder, J.G., Levi, S., On the search for Borel 1 selections, to appear.

[14] Ceder, J.G., Pearson, T.L., On typical bounded Darboux Baire 1 functions. Acta Math. Acad. Sci. Hungar 37(4), 1981, 339-348. MR82; 26005.

[15] Ceder, J.G.; Petruska, Gy., Most Darboux functions map big sets into small sets, to appear in Acta Math. Acad. Sci. Hungar.

[16] Farkova, J., About the maximum and the minimum of Darboux functions, Math Casopis 21 (1971) 110-116, MR46 #3704.

[17] Garg, K.M., Monotonicity, continuity and levels of Darboux functions, Colloq. Math. 28 (1973), 91-103. MR48 #2317.

[18] Jones, F. Burton; Thomas, E.S. Jr., Connected G_δ graphs, Duke Math. J. 33 (1966), 341-346. MR33 #702.

[19] Mustafa, Ibrahim, Ph.D. On upper-semi-continuous Darboux functions Thesis, University of California at Santa Barbara, in preparation.

[20] Neugebauer, C., Darboux functions of Baire class one and derivatives, Proc. Amer. Math. Soc. 13 (1962), 838-843. MR25 #1400.

[21] O'Malley, Richard J., Baire* 1 Darboux functions, Proc. Amer. Math. Soc. 60 (1976), 187-192. MR54 #5405.

[22] __________, Insertion of Baire* 1 functions, Rev. Roumaine Math. Pures Appl. 24 (1979), 1445-1448. MR81c : 26004.

[23] Petruska, Gy., An extension of the Darboux property and some typical properties of Baire-1 functions, Real Anal. Exchange 8 (1982-83), 62-64.

[24] Petruska, Gy.; Laczkovich, M., A theorem on approximately continuous functions, Acta Math. Acad. Sci. Hungar. 24 (1973), 383-387. MR48 #4217.

[25] Preiss, David, Limits of derivatives and Darboux-Baire functions, Rev. Roumaine Math. Pures Appl. 14 (1969), 1201-1206. MR48 #4397.

[26] __________, Maximoff's theorem, Real Anal. Exchange 5 (1979-80), 92-104. MR80m : 26007.

[27] __________, Algebra generated by derivatives, Real Anal. Exchange 8 (1982-83), 208-216.

[28] Pu, H.W.; Pu, H.H., Associated sets of Baire* 1 functions, Real Anal. Exchange 8 (1982-83), 479-485.

[29] Rinne, D., The cluster set structure of Darboux Baire 1 functions, to appear.

[30] Rinue D., Typical bounded functions in the Zahorski classes I, to appear.

[31] Rinue D., Typical bounded functions in the Zahorski classes II, to appear.

[32] Thomas, E.S. Jr., Some characterizations of functions of Baire class 1, Proc. Amer. Math. Soc. 17 (1966), 456-461. MR32 #6420.