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NEW RESULTS ON FUNCTION CLASSES INVARIANT UNDER CHANGE OF VARIABLE

Let X be a class of real functions with domain D, either an interval or the circle group. We say that X is <u>invariant under change of variable</u> if $f \in X$ implies $f \circ g \in X$ for every $g : D \to D$ homeomorphically. There are two principal ways in which such classes arise:

(i) A class of functions may be defined by an intrinsic property of the functions, such as a condition on the corresponding interval functions (if $I = (a,b)$, $f(I) = f(b) - f(a)$) which is independent of the length of intervals, or by invariant properties of the functions' level sets, such as a condition on the Banach indicatrix.

(ii) Given a property P, we may consider the class of functions f such that $f \circ g$ has property P for all admissible g.

Examples of the first type come readily to mind. The functions of bounded variation and most classes of functions of generalized bounded variation are of this type as is the class of functions f such that $\varphi(n(y;f)) \in L^p$, where φ is an increasing non-negative function on R^+ and $n(y;f)$ is the Banach indicatrix of f. Examples of the second type are easily produced. Consider those f such that the Fourier series of $f \circ g$ converges everywhere for each g, or those f such that $f \circ g \in L^1$ for each g.

A question which arises naturally here is: Given a class of functions with one of these types of characterizations, can we find a characterization of the other type?

Goffman and Waterman [7] characterized the class of continuous functions whose Fourier series converge everywhere for every change of variable. Baernstein and Waterman [2] characterized the analogous class for uniform convergence. It was believed [6] that the condition of Goffman and Waterman would suffice for functions that were <u>regulated</u> (have only simple discontinuities), but the suggested argument was inadequate. We have recently shown, however, that the condition is correct in this case [13]. We have also given new forms of the conditions for everywhere and uniform convergence [13] which make their relationship clear.

The condition of Goffman and Waterman (GW) is stated in terms of <u>systems of intervals</u> at a point. For each n, let I_{ni}, $i = 1,\dots,k_n$ be disjoint closed intervals indexed from left to right, i.e., I_n, $I_{n,i-1}$ is to the left of I_{ni}. Let there be a real x such that for every $\delta > 0$ there is an N such that $I_{ni} \subset (x,x+\delta)$ whenever $n > N$. Then the collection $\{I_{ni}\}$, $n = 1,2,\dots$, $i = 1,\dots,k_n$ is called a <u>right system</u> of intervals at x. A left system is defined similarly, $I_{ni} \subset (x-\delta,x)$ and indexed from right to left.

If I denotes an interval $[a,b]$, we write $f(I) = f(b)-f(a)$.

The *condition* GW *is*

$$\lim_{n \to \infty} \sum_{i=1}^{k_n} i^{-1} f(I_{ni}) = 0$$

for every right and left system.

In [7] it was observed that for continuous functions the requirement in GW that the I_{ni} be *disjoint* can be relaxed to that they be *nonoverlapping*. We have shown that this is true for *regulated* functions as well.

The condition UGW of Baernstein and Waterman which is necessary and sufficient for preservation of uniform convergence is that f *be continuous, satisfy* GW, *and*

$$\lim_{n \to \infty} \sum_{i=1}^{k_n} (k_n+1-i)^{-1} g(I_{ni}) = 0$$

for every system.

We have recently obtained the following simplification of the GW condition.

A regulated f *satisfies condition* GW *if and only if for every* $\epsilon > 0$ *and* x *there is a* $\delta > 0$ *such that for every finite sequence* $\{I_i\}$ *of nonoverlapping intervals indexed from left to right (right to left) in* $(x, x+\delta)$ $((x-\delta, x))$ *with* $\cup I_i \subset (x, x+\delta)$ $(\cup I_i \subset (x-\delta, x))$ *we have*

$$\left|\sum i^{-1} f(I_i)\right| < \epsilon .$$

We have also shown that UGW is equivalent to "GW uniformly in x" in the following sense.

f *satisfies the* UGW *condition if and only if for every* $\epsilon > 0$ *there is a* $\delta > 0$ *such that for every* x *and every finite sequence* $\{I_i\}$ *of nonoverlapping intervals indexed from left to right (right to left) with* $\cup I_i \subset (x, x+\delta)$ $(\cup I_i \subset (x-\delta, x))$ *we have*

$$\left|\sum i^{-1} f(I_i)\right| < \epsilon .$$

In the same paper [13] we made careful uniform estimates of the difference between a continuous function and the n-th partial sum of its Fourier series. Using these estimates in [14], we gave a new demonstration of the result of Baernstein and Waterman on preservation of uniform convergence.

Classes which seem closely related to those we have discussed are certain ΛBV spaces. If $\Lambda = \{\lambda_n\}$ is an increasing sequence of positive reals with $\Sigma\, 1/\lambda_n = \infty$, we say that $f \in \Lambda BV$ if sums of the form

$$\Sigma\, |f(I_n)|/\lambda_n$$

are uniformly bounded, $\{I_n\}$ being collections of nonoverlapping intervals in D. If $\lambda_n = n$, we say that f is of harmonic bounded variation (HBV). By X_C we will mean the continuous functions in class X. GW and UGW will denote the regulated functions satisfying GW and the continuous functions satisfying UGW respectively. It is known [10] that

$$HBV \subset GW \quad \text{and} \quad HBV_C \subset UGW .$$

We have asked if these inclusions are proper. Baernstein and Waterman showed that $GW_C \supsetneq UGW$. Then $GW_C \supsetneq UGW \supset HBV_C$, showing that

$$HBV \neq GW ,$$

which settles the first part of this question. Let OHBV be the class whose definition differs from that of HBV only in that the $\{I_n\}$ are ordered either from right to left or left to right. Belna showed [3] that

$$OHBV_C \supsetneq HBV_C$$

and we showed [12] that

$$OHBV_C \supsetneq UGW \supset HBV_C \quad \text{and} \quad OHBV_C - GW \neq \emptyset .$$

We have recently [15] improved this to

$$GW_C \supsetneq GW_C \cap OHBV_C \supsetneq UGW \supset HBV_C ,$$

but the second part of our question, whether this last inclusion is proper, has still to be answered.

Another problem we have considered recently is the preservation of the order of magnitude of the Fourier coefficients under a change of variable. For simplicity we assume here that f is regulated. Let $h(x)$ be a nondecreasing function which is concave downward. We wish to characterize the functions f such that, for admissible g,

$$(*) \qquad \widehat{f\circ g}(n) = O(h(n)/n) .$$

We had solved this problem for $h(n) = O(1)$. This class is the functions of bounded variation [11]. In general, if $\{I_i\}$ is a collection of nonoverlapping intervals, let

$$\sup\left\{\sum_1^n |f(I_i)| \,\Big|\, \{I_i\}_1^n\right\} = \nu(n;f) .$$

This has been called the <u>modulus of variation</u> by Čanturia [4], and he denotes the class of f such that $\nu(n;f) = O(h(n))$ by $V[h]$. Clearly this class is change-of-variable invariant. Čanturia has shown [5] that $\hat{f}(n) = O(\nu(n;f)/n)$, so we see that $f \in V[h]$ is sufficient in order that (*) holds. We have shown that this condition is also <u>necessary</u> [16].

Avdispahić [1] has shown that $V[h] \supset \Lambda BV$ if $\sum_1^n 1/\lambda_k = O(n/h(n))$. If $\sum_1^n 1/\lambda_k \simeq n/h(n)$, then Čanturia's estimate for $\hat{f}(n)$ if $f \in V[h]$ is the same as that of Wang [9] and Schramm and Waterman [8] for $f \in \Lambda BV$. We conjecture that except for the case $h(n) = O(1)$ in which $\Lambda BV = V[h] = BV$, we have

$$\Lambda BV \subsetneq V[h] .$$

In the case $\Lambda = H$ this may be verified, for

$$HBV \subsetneq OHBV \subset V[n/\log n] .$$

Another topic of recent investigation has been <u>Fourier effective</u> summability methods for ΛBV. A summability method is called Fourier effective for a class X if it sums the Fourier series of functions of that class to the value of the function a.e.. For regulated functions we require that the sum be $(f(x+)+f(x-))/2$ at every x. Convergence is Fourier effective for HBV but not for ΛBV if $\Lambda BV - HBV \neq \emptyset$ [10]. In general, we can define a Fourier effective method for ΛBV which is not effective for any $\Lambda' BV$ such that $\Lambda' BV - \Lambda BV \neq \emptyset$.

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