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MULTIPLIERS OF VARIOUS CLASSES OF DERIVATIVES

(Lecture presented at Real Analysis Symposium in Waterloo.)

Let f be a function (= mapping to $(-\infty,\infty)$) on the interval $J = [0,1]$ and let Φ be a system of functions on J. We say that f is a multiplier of Φ if and only if $f\varphi \in \Phi$ for each $\varphi \in \Phi$. The system of all multipliers of Φ will be denoted by $M(\Phi)$. If, e.g., Φ is closed under multiplication and if the function $\varphi(x) = 1 (x \in J)$ belongs to Φ, then, obviously, $M(\Phi) = \Phi$. It is well known, however, that derivatives behave badly with respect to multiplication. It is therefore of some interest to investigate the system $M(\Phi)$, if Φ is a "reasonable" class of derivatives.

Let D $[C, \Delta, C_{ap}]$ be the system of all finite derivatives [continuous functions, differentiable f., approximately continuous f.] on J. For each system Φ of functions on J let Φ^+ $[b\Phi]$ be the system of all nonnegative [bounded] elements of Φ.

R.J. Fleissner characterized in [1] and [2] the system $M(D)$. For this purpose he introduced the notion of a function of distant bounded variation. This notion can be defined in various ways. It seems that the simplest way is the following: Let f be a function on J. We say that f is of distant bounded variation if and only if

$$\limsup_{h\to 0+} \operatorname{var}(x+h, x+2h, f) < \infty \quad \text{for each} \quad x \in [0,1)$$

and

$$\limsup_{h\to 0+} \operatorname{var}(x-2h, x-h, f) < \infty \quad \text{for each} \quad x \in (0,1].$$

The first of these two conditions is, of course, equivalent to

$$\limsup_{n\to\infty} \operatorname{var}(x+\tfrac{1}{n}, x+\tfrac{2}{n}, f) < \infty \quad \text{for each} \quad x \in [0,1)$$

where n is an integer; similarly for the second.

If we denote by Y the system of all functions of distant bounded variation, we may express Fleissner's result by

$$M(D) = D \cap Y.$$

Fleissner posed in [1] the problem of characterizing the system $M(SD)$, where SD is the class of all summable (Lebesgue integrable) derivatives. This problem has been solved in [3]. Here I will formulate the corresponding result in a slightly different way.

If f is a function on an interval $[a,b]$ and if n is a natural number, let $v(n,a,b,f)$ be the least upper bound of all sums $\sum_{k=1}^{n} |f(y_k) - f(x_k)|$, where $a \le x_1 < y_1 \le x_2 < y_2 \le \cdots \le x_n < y_n \le b$. Let V be the system of all functions f on J such that

(1) $\limsup_{n\to\infty} v(n, x+\frac{1}{n}, x+\frac{2}{n}, f) < \infty$ for each $x \in [0,1)$

and

(2) $\limsup_{n\to\infty} v(n, x-\frac{2}{n}, x-\frac{1}{n}, f) < \infty$ for each $x \in (0,1]$.

It is obvious that $Y \subset V$. A solution of the mentioned problem is now given by the relation

$$M(SD) = D \cap V.$$

Another "natural" class of derivatives is D^+. The vector space E generated by it is, obviously, the system of all functions $f \in D$ such that $|f| \leq g$ for some $g \in D$ (so that, e.g., $bD \subset E$). To describe $M(E)$ we need the following notation. If f is a bounded nonnegative function on an interval $[a,b]$ and if r is a natural number, we set

$$A(r,a,b,f) = r^{-1} \sum_{k=1}^{r} \sup f([x_{k-1},x_k]),$$

where $x_k = a + k(b-a)/r$. Then $M(E)$ is the system of all bounded functions f on J such that

$$\lim_{r,n\to\infty} A(r,x,x+\tfrac{1}{n},|f-f(x)|) = 0 \quad \text{for each} \quad x \in [0,1)$$

and

$$\lim_{r,n\to\infty} A(r,x-\tfrac{1}{n},x,|f-f(x)|) = 0 \quad \text{for each} \quad x \in (0,1].$$

It is not difficult to prove that $M(D^+) = (M(E))^+$ and that

$$M(D) \subset M(SD) \subset M(E) \subset bC_{ap} \tag{3}$$

with proper inclusions. Further we have $M(D) \setminus C \neq \emptyset$, $\Delta \setminus M(D) \neq \emptyset$, $C \setminus M(SD) \neq \emptyset$. Some elements of $C \cap M(SD)$ are nowhere differentiable. These facts show that the role of continuity or differentiability in the investigation of multipliers is smaller than we might expect. We have, however, $\Delta \subset M(SD)$ and $C \subset M(E)$.

Let $f \in bC_{ap}$ and let T be the set of all points of discontinuity of f. If $f \in M(D)$, then T is finite. If $f \in M(SD)$, then T is countable and each nonempty subset of T has an isolated point; in particular, T is nowhere dense. If $f \in M(E)$, then T has measure zero (so that f is Riemann integrable). We see that the set of points of discontinuity of a function belonging to some of the first three systems in (3) is, in some sense, small. There is, however, a function $f \in M(D^+)$ such that $T \cap I$ is uncountable for each interval $I \subset J$.

Let Z be the system of all continuous functions of bounded variation on J. Since $Z \subset M(D)$ and $C \setminus M(SD) \neq \emptyset$, we see that neither of the first two systems in (3) is closed under uniform convergence. It can be shown, however, that the third is. Moreover, if $f \in M(E)$ and if φ is a function continuous on $(-\infty, \infty)$, then the composite function $\varphi \circ f$ belongs again to $M(E)$.

I would like to illustrate the situation by a few examples.

It is easy to construct a function $f \in bC_{ap}$ such that $f(0) = 0$, f is continuous on $(0,1]$ and that $f(2^{-n}) = 1$, $\mathrm{var}(2^{-n}, 2^{-n+1}, f) = 2$ for $n = 1,2,\ldots$. Then $f \in M(D) \setminus C$.

It is also easy to construct a function $f \in C$ for which (1) does not hold; then, of course, $f \in C \setminus M(SD)$.

Let S be the Cantor set. Let f be a function on J with the following properties: $f = 0$ on S; if $I = (a,b)$

is a component of $J \setminus S$, $b - a = 3^{-n}$, let $f = 0$ on $(a,\alpha] \cup [\beta,b)$, $f(c) = 1$ and let f be linear on $[\alpha,c]$ and on $[c,\beta]$, where $c = (a+b)/2 = (\alpha+\beta)/2$, $\beta - \alpha = 9^{-n}$. Then $f \in M(D^+)$ and f is discontinuous at each point of S.

Let $1 > a_1 > a_2 > \ldots$, $a_n \to 0$, $a_n/a_{n+1} \to 1$. There is a function $f \in C_{ap}$ such that $f(0) = 0$, f is continuous on $(0,1]$, $f(a_n) = 2$ for each n and $0 \leq f \leq 2$ on J. It is easy to construct a function $g \in D^+$ such that $g(0) = 1$ and that $fg \geq g$ on $(0,1]$. (Such a g may be continuous on $(0,1]$.) Since $(fg)(0) = 0$, we cannot have $fg \in D$. We see that the function f belongs to bC_{ap} and is Riemann integrable, but does not belong to $M(E)$.

Proofs of the above results will appear in Real Analysis Exchange.

REFERENCES

[1] R.J. Fleissner, Multiplication and the fundamental theorem of calculus: A survey, Real Analysis Exchange, Vol. 2, No. 1-1976, 7-34.

[2] _______________, Distant bounded variation and products of derivatives, Fund. Math. XCIV(1977), 1-11.

[3] Jan Mařík, Multipliers of summable derivatives, Real Analysis Exchange, Vol. 8, No. 2 (1982-83), 486-493.