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VARIATIONS ON BLUMBERG'S THEOREM

I. Introduction and Survey.

A "variant of Blumberg's theorem" is any theorem about relatively nice restrictions to relatively large sets for arbitrary functions from one space to another. In this paper we present some new results related to several variants of Blumberg's theorem, where the resulting restriction is either continuous or else "pointwise discontinuous" (abbreviated PWD). A function is PWD if it is continuous at each element of some dense subset of the domain of the function.

First we survey the history of four such variants of Blumberg's theorem, listed as Propositions A, B, C, and D below.

(A_Y) For every $f: X \to Y$, there exists $D \subset X$, D dense in X such that $f|D$ is continuous.

The case where Y = the reals, R, will be called Proposition (A). This is the original "Blumberg's theorem". H. Blumberg showed [5] that (A) holds for $X = R^2$, and he stated that his argument could be used to prove (A) holds for X = any complete metric space. Later [6], he proved (A_Y) holds for X and Y Euclidean spaces. Block and Cargal [4] showed that (A_Y) holds when X is a 2nd countable Hausdorff Baire space and Y is a 2nd countable Hausdorff space. Bradford and Goffman [8] showed that for metric spaces X, (A) holds if and only if X is a Baire space. White [32] extended this from metric X to topological spaces X which have σ-disjoint pseudobases (abbreviated "$\sigma\pi$-spaces"). This included a result of Bennett [2], [3], who showed that (A) holds when X is a regular semi-metrizable Baire space. Alas [1] showed that if X is a $\sigma\pi$ Baire space, then (A_Y) holds for every 2nd countable space Y (also see [22]). One advantage of knowing that (A_Y) holds for more general spaces Y (even if we are primarily interested in real valued functions) is that it then follows that if $f_1, f_2, \ldots$ is a sequence of real valued functions with domain such a space X, then there is a set $D \subset X$, D dense in X, such that for each i, $f_i|D$ is continuous. This result was proved by Haworth and McCoy in [19], and attributed to H. M. Schaerf. These results were also discussed and extended in [34], where White raised the question of whether it is true that if X is a space for which (A) holds, it necessarily follows that (A_Y) holds for every 2nd countable space Y. This question has been answered in the affirmative by Piotrowski and Szymanski [26].

The question of when (A_Y) holds for larger (non-2nd-countable) spaces has been studied. Letting the <u>weight, ωY</u> of a space Y be the minimum cardinality of a basis for Y, Bradford showed in [7] that if X and Y are metric spaces, then (A_Y) holds if and only if no open subset of X is the union of ωY or fewer nowhere dense sets (spaces with this latter property are called <u>ωY-Baire spaces</u>). We now know from the theorem of Stepanek and Vopenka [28] (also see [21]) that this places restrictions on the size of the range space in

Blumberg's theorem <u>in the metric case</u> because every open set in every metric space without isolated points is the union of c or fewer nowhere dense sets (c is the cardinality of R). However, these ideas have more general application in the topological case, as was shown by Szymanski in [29] (more results along this line should be forthcoming in [14]).

As for negative results, Blumberg pointed out in [6] that the set D cannot be made to have cardinality c even for $f: R \to R$ because of the function of Sierpinski and Zymund [27] which has no continuous restriction of cardinality c. Goffman [18] showed that even for 1-1 $f: R \to R$, the set D cannot necessarily be chosen so that $f|D$ is a homeomorphism, and Ceder showed [16] that even for $f: R \to R$, the set D cannot necessarily be chosen so as to make $f|D$ differentiable or monotonic.

There have been a number of papers investigating the difference between the Baire property and (non-metric) spaces for which (A) holds (of course, (A) $\Rightarrow$ Baire, even for general spaces). Levy [24] showed that (X Hausdorff Baire, linearly ordered) $\not\Rightarrow$ (A). White [32] showed (assuming the continuum hypothesis CH) that (X completely regular Hausdorff Baire) $\not\Rightarrow$ (A). Levy [25] showed that (X compact T_1) $\not\Rightarrow$ (A), and assuming $c = 2^{\Omega}$, where $\Omega = \omega_1$) that (X compact Hausdorff) $\not\Rightarrow$ (A). Finally, Weiss [30], [31] showed (under ZFC) that (X compact Hausdorff) $\not\Rightarrow$ (A), thus settling what had come to be known as "the Blumberg problem". See [33] for other examples of Baire spaces in which (A) fails.

Although the set D of Proposition (A) cannot be made to have large cardinality, this can be acheived if the property placed on $f|D$ is relaxed from continuity to pointwise discontinuity. To avoid double subscripts, we adopt the notational convention $\omega = \omega_0 = \aleph_0$, $\Omega = \omega_1 = \aleph_1$, and $c = 2^{\omega}$. Consider the following proposition:

(B_{Ω}) For every $f:X \to R$, there exists $W \subset Y$, <u>W Ω-dense in X</u>, such that $f|W$ is <u>PWD</u>.

There will be corresponding propositions (B_c) and (B_n) for every cardinal n, where the statement that <u>W is n-dense in X</u> means that every open subset of X intersects W in a set of cardinality at least n. The present author showed in [9] that (B_c) holds for $X = R$ or any complete separable metric space without isolated points (notice that this strengthens Blumberg's original theorem for these spaces), and an attempt was made to characterize the metric spaces in which (B_{Ω}) and (B_c) hold. The notion of a "Lusin set" is central in this investigation. A <u>Lusin set</u> is a set which has no uncountable nowhere dense in X subset, we also say such sets have <u>property L(rel X)</u>. For a given cardinal n, we say that a space has <u>property T_n</u> if no open subset of X can be written as a union $F \cup G$, where F is first category in X and every nowhere dense in X subset of G is of cardinality $< n$. We say X has property T_n^+ if no open subset of X can be written as a union $F \cup G$, where F is first category in X and every nowhere dense in X subset of G can be written as a countable union $N_1 \cup N_2 \cup \ldots$, where each

N_i is of local cardinality $< n$. It is shown in [9] that for metric spaces X, $T_\Omega \Rightarrow (B_\Omega)$ and $T_\Omega \Leftarrow (B_c)$, so, assuming CH, we have a characterization of the metric spaces in which $(B_\Omega) = (B_c)$ holds. Then, it is shown that for metric spaces X, $T_c^+ \Rightarrow (B_c)$ and that the implication is reversible if X is also separable. So we have a characterization of the separable metric spaces in which (B_c) holds. H. E. White substantially improved this situation in [35], where he characterized (under ZFC) the spaces in which (B_n) holds for $n = c$ and many other cardinals larger than n (of course for these larger cardinals, the spaces will necessarily be non-separable). A space X has property ωBn if no open subset of X can be written as a countable union $F_1 \cup F_2 \cup \ldots$, where each F_i has the property that for every every open set U, there exists an open $V \subset U$ such that every nowhere dense in X subset of $F_i \cap V$ has cardinality $< n$. White pointed out that for metric spaces X, $T_\Omega^+ \Rightarrow \omega B\Omega \Leftrightarrow T_\Omega$ and $T_c^+ \Rightarrow \omega Bc \Rightarrow T_c$ (all implications are reversible in separable metric spaces). White shows that for Hausdorff spaces, $\omega Bn \Rightarrow (B_n)$ for every cardinal $n > \Omega$, and that $\omega Bn \Leftarrow (B_n)$ if (1) $n = c$, (2) $cf(n) > c$, (3) $n > c$ and $cf(n) = \omega$, or (4) (assuming Martin's axiom) if $cf(n) \neq c$. "cf(n)" denotes the cofinality of n (see [17]).

As for negative results, it was shown (under CH) in [9] that even for $f: R \to R$, you can't necessarily make W 2nd category in R and have $f|W$ be PWD. It was shown (under CH) in [11] that even for $f:R \to R$, you can't necessarily make W (non-λ_0)-dense in R and have $f|W$ be PWD, where property λ_0 is the property of having Lebesgue measure zero, and the statement that <u>W is (non-λ_0)-dense in X</u> means that every open subset of X intersects W in a non-λ_0 set.

We now consider two propositions analagous to Propositions (A) and (B_n), where the requirement that the set D (or W) have the stated density relative to X is relaxed to require that it only have the stated density relative to itself.

(C) for every $f: X \to R$, there exists $D \subset X$, <u>D ω-dense in D</u>, such that $f|D$ is <u>continuous</u>.

This proposition has not been stated previously in this form, but it holds for a separable metric space X if and only if X is uncountable. Ceder proved a much stronger version of (C) in [16], where he showed that if X is an uncountable subset of R, then

(C+) for every $f: X \to R$, there exists $D \subset X$, <u>D bilaterally dense in D</u>, such that $f|D$ is <u>monotonic and "differentiable"</u>.

The quotation marks on "differentiable" indicate that $+\infty$ and $-\infty$ are allowed values of the derivative. There was an error in the monotonicicity part of his proof, but this was corrected in [20] and [10].

Finally, we consider the following PWD-variant of (C), where n is an arbitrary cardinal,

(D_n) for every $f: X \to R$, there exists $W \subset X$, <u>W n-dense in W</u>, such that $f|W$ is <u>PWD</u>.

This proposition has not been studied in general settings, but it was shown in [11] that if X is a subset of R, then the following proposition:

(D_c+) for every $f: X \to R$, there exists $W \subset X$,

<u>W bilaterally c-dense in W</u>, such that $f|W$ is <u>"differentiable" at each element of a dense in W subset D of W</u>,

holds for X if and only if X is non-$\sigma\nu_{c-}$ (defined below). We say that a set M has (1) <u>property c-</u> if it has cardinality less than c, (2) <u>property L_{c-}(rel X)</u> if every nowhere dense in X subset of M has property c-, (3) <u>property ν_{c-}</u> if it has property L_{c-}(rel M), and (4) <u>property $\sigma\nu_{c-}$</u> if it is a countable union of ν_{c-}sets. It is not possible to pick the set W of (D_c+) so that $f|W$ is also monotonic, and the question of whether W and D can be picked so that $f|D$ is monotonic is unanswered. In [11] it is shown that if $X \subset R$, then X is non-λ_0 if and only if it is true that for every $f: X \to R$, there exists $W \subset X$, <u>W (non-λ_0)-dense in W</u>, such that $f|W$ is <u>"differentiable" at each element of a dense subset D of W</u>.

II. Some New Results.

It is the primary purpose of this paper to present some results which represent progress in the direction of making the set W of propositions (B) and (D) large in some measure-theoretic or Baire-categoric sense. We have already pointed out that even for $f: R \to R$, the set W of (B) cannot be made (non- λ_0)-dense or (non-FC)-dense in R (assuming CH). <u>Property FC</u> is the property of being first category relative to the space X. Nevertheless, it is possible to make some progress in this direction.

The notions "λ_0", "L(rel X)", and "FC", of smallness of sets, fit into a hierarchy of "singularity properties" of subsets of separable metric spaces such as those discussed in Section 40 of Kuratowski's <u>Topology</u> Vol. I, and discussed in more detail in [12]. Of particular interest in this paper are those singularity properties listed below:

$$(\mathrm{I})\qquad \begin{tikzcd}[column sep=small, row sep=tiny] & & & \lambda_0 & \\ & L(\mathrm{rel}\ X) \arrow[r, Rightarrow] & U_0 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\ \omega \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & s^0 \arrow[r, Rightarrow] & TI \\ & S(\mathrm{rel}\ X) \arrow[r, Rightarrow] & AFC \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\ & & & FC & \end{tikzcd}$$

These properties are all notions of "smallness" defined for subsets of separable metric spaces which are generalizations of the notion of countability = property ω. For X = R, the sets with <u>property S(rel R)</u> are the so-called "Sierpinski sets", a set, every λ_0 subset of which is countable. M is "always first category" or has <u>property AFC</u> if it is true that for every perfect subset P of X, $P \cap M$ is first category relative to P. M is a "universal null set" or has <u>property U_0</u> (sometimes called "property β") if it is true that for every continuous Borel measure, μ, on X, the outer measure of M $\mu^*(M) = 0$. <u>Note</u>: The AFC $\Rightarrow$ FC implication requires that X have no isolated points, and the $U_0 \Rightarrow \lambda_0$ implication only has meaning if X is R or R^n for some n. M is a "Marczewski singular set" or has <u>property s^0</u> if it is true that for every perfect subset P of X, there exists a perfect subset Q of P such that $Q \cap M = \emptyset$. M is "totally imperfect" or has <u>property TI</u> if

M has no compact uncountable subset. There are many equivalent definitions of the above properties, as well as many related properties (e.g. "concentrated sets", "rarified sets", Borel's sets of "strong measure zero", etc.) which fit into the above diagram of implications and which are discussed in great detail in [12].

It is clear from the proof that $T_\Omega \Rightarrow (B_\Omega)$ given in [9] and the approach taken by White in [35] that the resulting set W is not only Ω-dense or (non-ω)-dense in X, it is actually (non-L(rel X))-dense in X. However, the set W is not necessarily (non-U_0)-dense in X. One might ask whether the set W of Proposition (B) can be made to be (non-U_0)-dense in X, or perhaps (non-AFC)-dense in X, provided the requirement placed on the space X is the proper strengthened version of property T_Ω? We will in fact see that for $X = R$, or any complete separable metric space without isolated points, the set W of Proposition (B_Ω) can actually be made to be (non-s^0)-dense in X. It will not be possible on the other hand to go so far as to make even the set W of Proposition (D_Ω) (non-TI)-dense in X because of the existence of functions $f\colon R \to R$ which transform every perfect set into R (see Ex. 3.3 of [15]).

We have already pointed out that CH implies that even for $f\colon R \to R$, you cannot make the set W of Proposition (B_Ω) either (non-λ_0)-dense or (non-FC)-dense in R, nor is it possible to make the set W of Proposition (D_Ω) (non-FC)-dense in W.

In order to obtain the desired variations on Propositions (B_Ω) and (D_Ω), we let "property P" represent an undefined singularity property which prescribes a σ-ideal of subsets of a space X (as do properties ω, U_0, AFC, λ_0, s^0, and FC of (I)). We say a set M has property $L_P(\text{rel } x)$ if every nowhere dense in X subset of M has property P. M has property ν_P if it is $L_P(\text{rel } M)$, and property $\sigma\nu_P$ if it is the countable union of sets with property ν_P. Using arguments similar to those of [9], [10], and [11] we obtain the following two theorems.

Theorem 1: If X is a separable metric space, then

(B) for every $f{:}X \to R$, there exists $W \subset X$, W (non-P)-dense in X such that $f|W$ is PWD

holds if and (under CH) only if

(B') no open subset of X is the union of an FC set and an $L_P(\text{rel } X)$ set.

Theorem 2: If X is a separable metric space, then

(D) for every $f{:}\ X \to R$, there exists $W \subset X$, W (non-P)-dense in W such that $f|W$ is PWD

holds if and (under CH) only if

(D') W is non-$\sigma\nu_P$.

If $P = \omega$, then $L_P(\text{rel } X)$ is just $L(\text{rel } X)$, ν_P is just property ν (of [23]), and $\sigma\nu_P$ is the property referred to as "property L_1" in [10] and [13]. For $P = \omega$, Theorem 1 includes the main theorems of [9], and Theorem 2 is a new theorem. Theorems 1 and 2 have separate interpretations when property P is taken to be either U_0, AFC, λ_0, s^0, or FC.

The idea of starting with a singularity property P (other than $P = \omega$) and then considering the "Lusin generalizations",

L_P(rel X), ν_P, and $\sigma\nu_P$, of that property was investigated in some detail in [13]. The cases where $P = U_0$, AFC, λ_0, s^0, or FC were not considered in [13] and are discussed briefly below. Let us say that a set has property (FC) v (L_P) if it is the union of an FC set and an L_P(rel X) set.

1. For $P = U_0$, properties L_P(rel X), ν_P, and $\sigma\nu_P$ are all the same as U_0. Property (FC) v (L_P) is more general than either property U_0 or FC, but restrictive enough that no open set in a perfect Polish space has property (FC) v (L_P). Thus, Theorems 1 and 2 would have applications for $P = U_0$.

2. For P = AFC, properties L_P(rel X), ν_P, $\sigma\nu_P$, and (FC) (L_P) are successively more and more general, each implying the next, and (under CH) are all different. No open set in a perfect Polish space has property (FC) v (L_P), so both Theorems 1 and 2 have applications for P = AFC.

3. For $P = \lambda_0$, properties L_P(rel X), ν_P, and $\sigma\nu_P$ are all the same as λ_0, and Theorem 2 has application to subsets X of Euclidean spaces. But every subset of any subspace of a Euclidean space would have property (FC) v (L_P), so Theorem 1 would have no application for $P = \lambda_0$.

4. For $P = s^0$, properties L_P(rel X), ν_P, and $\sigma\nu_P$ are all the same as s^0. Property (FC) v (L_P) is more general, but no open set in a perfect Polish space has this property. Thus, both Theorems 1 and 2 have application for $P = s^0$. The fact that (B) holds for $X = R$ and $P = s^0$ yields a strong improvement of Blumberg's original theorem.

5. For $P = FC$, it follows that every subset of any space would have property $L_P(\text{rel } X)$, so neither Theorem 1 nor Theorem 2 have application for $P = FC$.

As a final comment, we remark that while it is not possible to get a "homeomorphism variant" of Proposition (A), it is possible to get one for Proposition (C).

Theorem 3: If X and Y are uncountable separable metric spaces and $f: X \to Y$ is 1-1, then there exists $D \subset X$, D dense in D, such that $f|D$ is a homeomorphism.

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