J. Aczél, Department of Pure Mathematics, University of Waterloo,
Waterloo, Ontario N2L 3G1

On Locally Bounded Maps of Sequence Spaces

Dedicated with warm friendship to Professor Jan Mikusinski on his 70th birthday and 40th scientific anniversary.

In an interesting paper Jurkat and Shawyer (1981) state the following.

Let A and B be linear summation methods for (complex) sequences (the respective 'sums' being denoted by $\lim_A$ and $\lim_B$, respectively), such that at least the constant sequence $\{1\}$ and the alternating sequence $\{(-1)^n\}$ are both A- and B-summable. A (complex) function f has the property P if, for every A-summable sequence $\{x_n\}$, the sequence $\{f(x_n)\}$ is B-summable and

(1) $$\lim_B f(x_n) = f(\lim_A x_n).$$

'Theorem' 1 (Jurkat-Shawyer 1981). *Let* $f: \mathbb{C} \to \mathbb{C}$ *be continuous at at least one point. Let*

(2) $$\lim_A\{1\} = \lim_B\{1\} = 1$$

and

(3) $$\lim_A\{(-1)^n\} = \lim_B\{(-1)^n\} = \theta \in \mathbb{C} \qquad (\theta\neq 1,\ \theta\neq -1).$$

Then f *has the property* P *if, and only if, there exist complex constants* a,b,c *such that* $f(z) = az + b\bar{z} + c$ *for all* $z \in \mathbb{C}$.

In this note we make a slight correction and generalize this theorem somewhat.

The correction concerns the "if" part. The conjugate $f(z) = \bar{z}$ is certainly of the required form $(a=c=0, b=1)$. But, for this f, we have

$\{f((-1)^n)\} = \{(-1)^n\}$ so, by (1) and (3),

$$\theta = \lim_B\{-1\}^n = \lim_B\{f((-1)^n)\} = f(\lim_A\{(-1)^n\} = f(\theta) = \overline{\theta}.$$

So $f(z) = \overline{z}$ or <u>any</u> $f(z) = az + b\overline{z} + c$ <u>with</u> $b \neq 0$ <u>has the property</u> P <u>only if</u> θ <u>is real</u>.

Well this is not so unexpected. Actually, it would be strange if the A- (or B-) limit of $\{(-1)^n\}$ were not real. But it is nice that $\theta = \overline{\theta}$ actually follows from such minimal suppositions.

In fact, the supposition (3) that the A-limit and the B-limit of $\{(-1)^n\}$ are equal, may not seem so natural (if we do not suppose that both are 0). So let us suppose, instead of (3),

$$(4) \qquad \lim_A\{(-1)^n\} = \theta, \qquad \lim_B\{(-1)^n\} = \omega.$$

With a further stretch of imagination we may replace also (2), for our money, by the more general

$$(5) \qquad \lim_A\{1\} = \phi, \qquad \lim_B\{1\} = \psi.$$

Again, we will suppose that the absolute values of the (A-,B-) limits of these two sequences are different:

$$(6) \qquad \theta \neq \pm\phi, \qquad \omega \neq \pm\psi.$$

Further, it may seem slightly more natural to suppose f to be <u>bounded</u> (in the real case from one side) on a small interval or even on a set of positive measure rather than continuous at a point (continuity at a point does, of course, imply local boundedness). We will prove the following.

Theorem 2. Let A and B be linear summation methods satisfying (4) and (5) with (6), let $f: \mathbb{C} \to \mathbb{C}$ be bounded on a set of positive measure, and let f satisfy (1). Then there exist complex constants a,b,c such that $f(z) = az + b\bar{z} + c$ for all $z \in \mathbb{C}$. Conversely, if $f(z) = z$, $f(z) = \bar{z}$ or $f(z) = 1$ are solutions of (1), then, in (4) and (5), we have $\phi = \psi$, $\theta = \omega$ or $\phi = \bar{\psi}$, $\theta = \bar{\omega}$ or $\psi = 1$, respectively. If, and only if,

$$\phi = \psi = 1, \quad \theta = \omega \in \mathbb{R},$$

then every function of the form $f(z) = az + b\bar{z} + c$ is a solution of (1).

So, if the most obvious functions $(z, \bar{z}$ and $1)$ are to have property P, then necessarily $\lim_A\{1\} = \lim_B\{1\} = 1$ and $\lim_A\{(-1)^n\} = \lim_B\{(-1)^n\}$ is real, which is as it should be.

Jurkat and Shawyer (1981) gave an example showing that the first part of the theorem does not hold if (6) is not satisfied.

The proof of the first part of Theorem 2 goes along the lines given by Jurkat and Shawyer (1981). Take the sequences

$$\{x_n\} = \{t,u,t,u,\ldots\} = \frac{t+u}{2}\{1\} + \frac{t-u}{2}\{(-1)^n\},$$

$$\{f(x_n)\} = \{f(t),f(u),f(t),f(u),\ldots\} = \frac{f(t)+f(u)}{2}\{1\} + \frac{f(t)-f(u)}{2}\{(-1)^n\}.$$

By the linearity of A- and B-summability we see that these sequences are A- resp. B- summable for all t,u and, with

(7) $\alpha = (\phi+\theta)/2, \qquad \beta = (\phi-\theta)/2, \qquad \gamma = (\psi+\omega)/2, \qquad \delta = (\psi-\omega)/2,$

we get

$$\lim_A\{x_n\} = \frac{t+u}{2}\phi + \frac{t-u}{2}\theta = \alpha t + \beta u,$$

$$\lim_B\{f(x_n)\} = \frac{f(t)+f(u)}{2}\psi + \frac{f(t)-f(u)}{2}\omega = \gamma f(t) + \delta f(u)$$

and, by (1),

$$f(\alpha t+\beta u) = \gamma f(t) + \delta f(u) \tag{8}$$

for all $t,u \in \mathbb{C}$, where, because of (6) and (7), $\alpha\beta\gamma\delta \neq 0$. But every solution, bounded on a set of positive measure, of this functional equation is of the form (cf. Aczél, 1966, sections 2.2.6 and 5.1.1)

$$f(z) = az + b\bar{z} + c. \tag{9}$$

(This result is not true if at least one of $\alpha,\beta,\gamma,\delta$ is 0. This shows the importance of condition (6), which, however, is also natural in the setting of our problem: The A- and B-limits of $\{(-1)^n\}$ are not equal to the A- resp. B-limits of $\{1\}$ or $\{-1\}$.)

Now substitute (9) back into (8)

$$a\alpha t + a\beta u + b\bar{\alpha}\bar{t} + b\bar{\beta}\bar{u} + c = \gamma at + \gamma b\bar{t} + \gamma c + \delta au + \delta b\bar{u} + \delta c. \tag{10}$$

Comparing coefficients and taking (7) into consideration, we have that,

if $a \neq 0$, then $\alpha = \gamma$, $\beta = \delta$, that is, $\phi = \psi$, $\theta = \omega$,

if $b \neq 0$, then $\bar{\alpha} = \gamma$, $\bar{\beta} = \delta$, that is, $\bar{\phi} = \psi$, $\bar{\theta} = \omega$,

and, if $c \neq 0$, then $\gamma + \delta = 1$, that is, $\psi = 1$.

If all these conditions are satisfied, then (10) holds identically. □

In exactly the same way we can prove the following.

Theorem 3. Let A and B be linear summation methods on real sequences satisfying (4) and (5) with (6) (θ,ω,ϕ,ψ real), let $f: \mathbb{R} \to \mathbb{R}$ be bounded from one side on a set of positive measure and let f satisfy (1). Then there exist real constants a,c such that $f(x) = ax + c$ for all $x \in \mathbb{R}$. Conversely, if $f(x) = x$ or $f(x) = 1$ are solutions of (1), then, in (4) and (5), $\phi = \psi$, $\theta = \omega$ and $\psi = 1$, respectively. If the shift $f(x) = x+1$ is a solution, then $\phi = \psi = 1$, $\theta = \omega$. Moreover, if, and only if, $\phi = \psi = 1$, $\theta = \omega$, then every real function of the form $f(x) = ax + c$ is a solution of (1).

Boundedness from one side on a set of positive measure is a rather weak regularity condition. But the proof does not work if no regularity is supposed at all. As was shown by Losonczi (1964), if no regularity of f is required, then not only does (8) (say, in the real case) have solutions different from $f(x) = ax + c$, but not even $\alpha = \gamma$, $\beta = \delta$ follows in every case when (8) has nonconstant solutions.

References

J. Aczél (1966), Lectures on Functional Equations and Their Applications. Academic Press, New York-London.

W.B. Jurkat and B.L.R. Shawyer (1981), On continuous maps of sequence spaces. Analysis 1, 209-210.

L. Losonczi (1964), Bestimmung aller nichtkonstanten Lösungen von linearen Funktionalgleichungen. Acta Sci. Math. (Szeged) 25, 250-254.