Ralph Henstock, Department of Mathematics, New University of Ulster, Coleraine, Co. Londonderry, N. Ireland.

THE LEBESGUE SYNDROME

First I would like to express my thanks to George Cross for inviting me to take part in this Symposium on Real Analysis. Then I should explain that, being one whose research has been mainly in the theory of integration, my talk is partly historical, partly personal, and partly a looking to the future.

Good accounts of the origins of present-day mathematics have been given, and, in particular, for integration theory there are two articles by T.H. Hildebrandt (-1980) [21],[22], and two by P. S. Bullen [1],[2]. The latter two can be used to supplement the given references, which are the bare minimum for this paper. For the earlier period I. Grattan-Guinness [16] and J. V. Grabiner [15] will be found most useful.

In brief, for about 150 years from the time of I. Newton (1642-1727) and G. W. Leibnitz (1646-1716) the word _function_ had a rather badly defined sense. It was usually a variable y connected to a variable x by an equation involving a finite number of symbols of algebraic operations (addition, subtraction, multiplication, division, and extraction of roots), trigonometric operations (using sin, cos, tan, arcsin, arccos, arctan) and logarithmic and exponential operations (using $\log_e$ and e^x), the so-called _elementary functions_ of G. H. Hardy (1877-1947), see [17]. Lebesgue (1875-1941) in [26] seems to call them _eulerian continuous_, and mentions that a great number of functions expressed in this way, have integrals that can also be expressed in this way. Hardy [17] goes into various questions of this type. But Lebesgue [26] points out that if $f(x) \geq 0$ and is bounded and eulerian continuous, and if A(x) is the area under the curve $y = f(x)$ from a to x, then $dA/dx = f(x)$. The relation between A(x) and x is clearly a good geometric relation and A(x) is a decent function, but A(x) need not be an elementary function. Further, J.B.J. Fourier (1768-1830) showed that trigonometric series, that can represent eulerian continuous functions, can also represent discontinuous functions formed of parts of functions. An easy example is

$$f(x) = x \ (0 \leq x < \pi), \ -x(-\pi < x < 0),$$

and people used to regard this as two functions, not one. Some students at an elementary level think the same even today. Yet we can write it as $+\sqrt{(x^2)}$, so that even this is an elementary function. Another example is

$$f(x) = \tfrac{1}{2} \ (x = 0), \ 0 \ (0 < x < \pi), \ \tfrac{1}{2} (x = \pi), \ 1 \ (\pi < x < 2\pi)$$

with f(x) having a period 2. The Fourier series of this f(x) has sum equal to f(x) for all x. Examples like this showed the necessity of extending the idea of a function. The first to give the modern definition was A. L. Cauchy (1789-1867), but even Cauchy said that functions were given by analytic expressions. G. F. B. Riemann (1826-1866) took Cauchy's definition in full generality, omitting the requiremen of analytic expression. If anyone wishes to look at the origins of modern analysis, one need not look any earlier than Cauchy, he was the Euclid of analysis, unless one wants to stride into the morass of ill-defined ideas and illogical reasoning. Yet much good mathematics was produced before Cauchy.

People turned to integration for a variety of reasons and needs. Nowadays they can be roughly classified into four kinds. First are appropriate integral formulae for areas on flat and curved surfaces, volumes of solid figures, lengths of certain curves, and measures of mass, moments of inertia, charge given a charge density, magnetism given a magnetic density, and so on. Secondly, integrals are needed to solve differential and integral equations in the sciences and elsewhere. Next, continuous linear functionals and self-adjoint (hypermaximal) linear operators over spaces of functions often use Stieltjes-type integrals in functional analysis. This is of course a modern use. Finally, integrals are used for statistical distributions and are often used to approximate sums, such as normal integrals for binomial sums. If anyone knows of another field where integrals are used, please let me know.

Having set the scene for their use, we now turn to the historical introduction of integrals, beginning with Cauchy. Leibnitz's notation for integrals used a large S, while Fourier used the modern notation $\int_a^b f(x)dx$. Cauchy defined this definite integral on the finite interval $[a,b]$ of the real line by using divisions

$$a = x_0 < x_1 < \ldots < x_n = b \quad (x_j - x_{j-1} < \delta,\ j = 1, \ldots, n)$$

He then took the limit as $\delta \to 0$ of sums

$$\sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1})$$

while Riemann replaced $f(x_{j-1})$ by $f(\xi_j)$, where each ξ_j takes every value in $[x_{j-1},x_j]$, to define his integral. Then J. G. Darboux (1842-1917) used the infimum $m(x_{j-1},x_j)$ and supremum $M(x_{j-1},x_j)$ of f in $[x_{j-1},x_j]$ in place of $f(\xi_j)$ in the sums. If the sums using m and the sums using M tend to the same limit as $\delta\to 0$ we have the Riemann-Darboux integral. It is easy to show that the Riemann and Riemann-Darboux integrals are equivalent when the values of f are real. Note also that we could assume that $\xi_j = x_{j-1}$ or $\xi_j = x_j$, not necessarily the same end for each interval, for each time that $x_{j-1}<\xi_j<x_j$ we can put

$$f(\xi_j)(x_j-x_{j-1}) = f(\xi_j)(\xi_j-x_{j-1})+f(\xi_j)(x_j-\xi_j)$$

and the new sum has the ξ_j at the ends of the intervals.

At about that time many new functions were produced, and S. Saks [31] gave typical cries from older analysts. H. Poincaré (1854-1912) said in French something like this. 'Previously, when one invented a new function it was for practical purposes. Nowadays one invents them intentionally to put at fault the reasonings of our fathers and one can never get away from that!' Ch. Hermite (1822-1901) said, 'I turn myself with terror and horror from that lamentable wound of functions that have no derivatives'. It could have been the Fourier series $\sum_{n=0}^{\infty} a^n\cos(b^n\pi x)$ for $ab>1$ with b an odd integer and $0<a<1$. I wonder what they would have said about the example of W. Sierpiński (1882-1969) in [32] of a plane set that has one point only, on each line parallel to the x-axis and on each line parallel to the y-axis, and yet that is Lebesgue non-measurable and so Jordan (1838-1922) non-measurable!

(Jordan measure corresponds to Riemann integration). E. Borel (1871-1956) produced his theory of measure, and R. Baire (1874-1932) his transfinite sequences, and then at the turn of the century came H. Lebesgue and his integral. Someone once said that Borel could have kicked himself, for not having produced Lebesgue's integral before Lebesgue. Whether that was true or false, Lebesgue's work was the culmination of all that had gone before. There are 57 varieties of definition of Lebesgue integration, Lebesgue himself divided up the range of values of f(x), supposed bounded by A and B, $A\leq f(x)<B$, and first found the measure $\mu(y_{j-1},y_j)$ of the set of x where $y_{j-1}\leq f(x)<B$, taking $A = y_0<y_1<\ldots<y_n = B$. Then he found the limit of sums

$$y_{j-1}\mu(y_{j-1},y_j)$$

as the greatest of the y_j-y_{j-1} tends to 0. He could have omitted A and B and have used infinite sums, to cover the case when f is unbounded, for in the absolute case everything converges. Lebesgue's integral was a giant step forward, and mathematicians voiced their acclaim. W. H. Young (1863-1942) had his own method, using monotone sequences repeatedly, his method now going under the later name of P. J. Daniell (1889-1946) [6], but Young enthusiastically took up Lebesgue's method Many years ago Mrs. Tanner (Miss R. C. Young) gave me many of her father's papers, and I have photocopied some more, and in his papers [35],[36], have found that he had written the papers before reading Lebesgue's work just before March 16th, 1904. In subsequent papers he mentioned Lebesgue's papers with enthusiasm. Again, last year Jean Mawhin gave me de la Vallée

Poussin's (1866-1962) papers on integration, and his book [7], which I had read avidly when beginning research. There is a marked difference between his papers before 1900 and those after 1900, Lebesgue's influence was clearly of a major importance in his case. Interestingly, the papers before 1900 contain results for limits of Riemann integrals that cannot be included in Lebesgue integration theory. However, there are exceptions to every rule, G. H. Hardy lived through that period but ignored Lebesgue's work. We are now 80 years further on, and yet Lebesgue still has a very powerful influence on research workers in integration theory, as one can see in Mathematical Reviews, section 28.

It soon became clear that, however good the Lebesgue integral appeared to be, it was not good enough to integrate all derivatives, and A. Denjoy (1884-1974), [8],[9], gave a construction in 1912 that began with the Lebesgue integral over some intervals and ended with the totalisation or what is now known as the Denjoy-Perron integral. O. Perron (1880-1975, [30], in 1914 gave his own integral using derivatives, that was proved equivalent to Denjoy's first integral. Later it was found that not all everywhere convergent trigonometric series have sum functions integrable by those methods, and one has to go even further, to Burkill's Cesàro-Perron integral [3], his symmetrical Cesàro-Perron integral [4], and the Marcinkiewicz-Zygmund integral [29].

Everyone researching in the theory of integration will have his or her own description of the original attraction to it. The personal side of this talk may explain mine. I was brought up in the Lebesgue tradition, having lectures in 1942 from J. C. Burkill who followed W. H. Young. Next were lectures from R. G. Cooke (1895-1965) in 1943 who followed Titchmarsh's (1899-1963) book [33], and finally lectures from A. S. Besicovitch (1891-1979) in 1947 who in the main followed Saks [31], except for his own differentiation theory. Also, from 1943 on, I read Saks [31] and de la Vallée Poussin [7]. I always regret having no time in 1947 to attend Besicovitch's lectures on non-absolute integration in the Denjoy way, but had to squash a year's lectures into 6 months. Having had R. G. Cooke's lectures on infinite matrices and the summability of series in 1943, when it came to part-time research in 1944 under Paul Dienes (1882-1952), while working in the Ministry of Supply, I suggested doing the analogue for integrals of the summability theory of Bosanquet for sums. Here, the necessary and sufficient conditions on (a_{mk}) in order that $\sum_{k=1}^{\infty} a_{mk} b_k$ should exist for $m = 1,2,\ldots$ and should tend to $\sum_{k=1}^{\infty} b_k$ as $m \to \infty$ whenever the last sum is convergent, are that $\lim_{m\to\infty} a_{mk} = 1$ $(k = 1,2,\ldots)$ and that for some fixed finite M independent of m,

$$\sum_{k=1}^{\infty} |a_{m,k+1} - a_{mk}| \leq M \quad (m = 1,2,\ldots).$$

Anyone with half an eye for the integration will see that the last condition would in the analogue produce a function of bounded variation. I suggested this as a research problem, but Dienes said, 'I have had <u>so</u> many students doing summability. <u>You</u> will do integration'. It is such a good tale of the intransigence of a supervisor that it is a pity to say that he added, 'Do you think it a good idea?' Not knowing of the research he had in mind, naturally I said, 'Yes'. So one who had scorned Riemann theory in favour of Lebesgue, had to examine Riemann sums with a norm or mesh limit, or a limit by refinement of subdivisions, and it was an easy progression to Burkill's interval functions, with a few Lebesgue problems, while still wrestling with that first research problem. Finally it came out and was published in 1955, 11 years afterwards. In between, for example, L. Tonelli's (1885-1946) definition [34] of Lebesgue integration was examined. For every measurable function f in [0,1] there is an open set G of measure less than a given $\varepsilon>0$, such that f is continuous in $[0,1]\setminus G$. Continuing f by a line across each component interval of G we see that f is the limit almost everywhere of a sequence of continuous functions f_n. Tonelli's definition is the limit of the integral of f_n, which works when f is Lebesgue integrable. Approximate to the integral of f_n by a Riemann sum and then with care a sequence of Riemann sums tends to the Lebesgue integral. One needs to tidy up as the parts of the Riemann sums in the open sets might not use the values of f. But in 1948 I had no means of specifying the sums and so dropped the matter. Then the first problem clarified. I needed $\int_0^\infty f dg$ for all bounded Baire functions f, wishing to prove that g is of bounded variation on each finite range. It was no use assuming this <u>a priori</u>, so Lebesgue or J. Radon (1887-1956) integration was out. Ward integration also proved useless as I needed g to be <u>a priori</u> any bounded Baire function, and when g is the characteristic function of the rationals and the integral exists as a Ward integral even over a finite interval, then f is constant there. To prove this result I used what was later called the Riemann-complete integral, without realising all the implications. This was paper [18], pp. 277-278, proof of Theorem 1, in 1955, and I used it again in 1957 in [19], pp. 97-98, Theorem 1. To finish the first problem I integrated by parts in the Ward integral, so getting a strong enough integral and the finish of the proof. The next topic was to put the convergence factors into the <u>definition</u> of the Ward integral, like Burkill's Cesàro-Perron integrals, to give N-integration. The introduction to the original paper had another definition called the N-variational integral, just to try it out, and I expected the Referee to remove it. I inferred afterwards that the Referee was Miss W. L. C. Sargent (1905-1979); she said, 'I think the author would do well to lay much more stress on his descriptive definition and to explain the definition more carefully. This seems to involve a new idea for a descriptive definition, and there is a danger it will tend to be overlooked because of the heavy working of the rest of the paper. It might in fact be a good plan either to separate the paper into two, or to write a short note on the descriptive definition of the Ward integral to be published separately'. Being so encouraged by that perceptive Referee, I carried out both suggestions, publishing three papers. Good advice is not always wasted! Axioms being needed for an abstract space, one axiom led me to the Riemann integral using complete sets, or the Riemann-complete integral, published 1961. Thus the journey into convergence-factor integration theory was reversed, returning to Riemann sums but with a more general limit. This work was independent of J. Kurzweil's paper [23] in 1957, and I was informed of that paper by K. Kartàk in a letter dated 3rd October 1963. So, as often occurs in mathematics, it could be a matter of argument who has the priority. In any case, Lebesgue [25], pp. 30-33, showed that his integral is the limit of Riemann sums, and so did Denjoy [12], [13], though neither gave anything explicit. The first explicit construction was in Beppo Levi (1875-   ), [27], in 1941, and earlier in 1923, according to Mrs. Foglio. It was a construction of the measure of the set of ordinates of the function to be integrated, assumed non-negative. As for me, I was turned upside down in 1958, throwing away Lebesgue and grasping Riemann! The Riemann-complete integral (now called the generalized Riemann integral and based on division spaces) is so linked with the calculus that it is more practical than Lebesgue's integral. Also it is a non-absolute integral while Lebesgue's is an absolute integral, the difference being analogous to the difference between all convergent series and absolutely convergent series.

The number of papers on Lebesgue measure and integration is still large, for instance, in the last year section 28 of Mathematical Reviews contains 195 or so reviews, excluding those on ergodicity, with some more in section 26. There are at least 19 papers on set-valued measures, 16 on the extension of measures and similar set functions, 11 on product measures, 9 on fuzzy measures and integrals, 6 on inequalities, 5 on the Radon-Nikodym theorem, and so on. In that year only

6 papers reviewed are devoted to non-absolute integration even though the theory is easier than most Lebesgue theories if the generalised Riemann or variational integral is used. It seems that analysts act towards integration in a way analogous to those who only use absolutely convergent series and ignore conditionally convergent series, and you can imagine what an uproar would occur if the experts in series behaved in that way! In fact you have the Lebesgue syndrome if you ignore the work done on non-absolute integration. It is 80 years since Lebesgue made his vast improvement, and nearly 30 years since the non-absolute theory was simplified. May I throw out a challenge to you to look at the papers using Lebesgue theory, to see whether the proofs can be improved and the contents generalized by generalized Riemann methods. The good work produced by R. L. Jeffery (-), R. D. James (1909-1979), and the analysts at this Seminar, must be built upon and carried forward. Even the simple device of including the dates of dead integration experts gives a feeling for the history and a pointer for the future in showing what has been done and what is left to be done. As for me, in the next year there is work on stochastic and other functional integrals, and work on martingales using division space methods. A question of Pfeffer on the integrability of $f\phi$ on $[a,b] \times [\alpha,\beta]$ when

$$\int_a^b f dx \quad , \quad \int_\alpha^\beta \phi d\xi$$

exist as Perron integrals, has been answered and extended to Çesàro-Perron and similar integrals, but not to the James integrals. For some years, Mrs. Foglio has asked me for a

G. Fubini (1879-1943) theorem for Cesàro-Perron integrals and now one may appear. Other general questions can be asked, for instance, why is one convergence-factor integral more powerful than another? Is there a better limit theorem than the Arzelà-Lebesgue dominated convergence theorem and its extensions to non-absolute integration? So much needs to be done, so that I should stop and let us get on with the task.

REFERENCES

[1] P. S. Bullen, 'Axiomatisations of various non-absolute integrals', Proc. Southeast Asian Bull. Math. 1979, special issue b. pp. 173-189, MR 81i:26009.

[2] __________, 'Non-absolute integrals: a survey', Real Anal. Exchange 5(1979-80) 195-259, MR 82a:26006.

[3] J. C. Burkill, 'The Cesàro-Perron integral', Proc. London Math. Soc. (2) 34 (1932) 314-322, Zbl. 5. 392.

[4] __________, 'Integrals and trigonometric series', Proc. London Math. Soc. (3) 1 (1951) 46-57, MR 13, 126.

[5] A. L. Cauchy, Oeuvres.

[6] P. J. Daniell, Ann. of Math. (2) 19 (1917/18) 279-294; ibid. (2) 22 (1919/20) 203-220; Jbuch 47, 911.

[7] De la Vallée Poussin, Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire (Paris, deuxieme édition, 1934), Zbl. 9. 206.

[8] A Denjoy, 'Une extension de l'intégrale de M. Lebesgue', Comptes Rendus Acad. Sci. Paris Sér. A-B 154 (1912) 859-62.

[9] __________, 'Calcul de la primitive de la function dérivée la plus générale', Comptes Rendus Acad. Sci. Paris Sér. A-B 154 (1912) 1075-78.

[10] _________, 'Sur la dérivation et son calcul inverse', Comptes Rendus Acad. Sci. Paris Sér. A-B 162 (1916) 377-80.

[11] _________, 'Mémoire sur la totalisation des nombres dérivée la plus générale', Ann. Sci. École Norm Sup. (3) 33 (1916) 127-222; ibid. 34 (1917) 181-238.

[12] _________, 'Sur l'intégration riemanienne', Comptes Rendus Acad. Sci. Paris Sér. A-B 169 (1919) 219-220.

[13] _________, 'Sur la définition riemannienne de l'intégrale de Lebesgue', Comptes Rendus Acad. Sci. Paris Sér. A-B 193 (1931) 695-98, Zbl. 3. 106.

[14] ________, Leçons sur le calcul des coefficients d'une série trigonométrique (Paris, 1941, 1949), MR 8, 260 and MR 11, 99.

[15] J. V. Grabiner, 'Who gave you the epsilon? Cauchy and the origins of rigorous calculus', American Math. Monthly 90 (1983) 185-94, MR.

[16] I. Grattan-Guinness, The development of the foundations of mathematical analysis from Euler to Riemann, The M.I.T. Press, Cambridge, Massachusetts 1970, MR 58# 15948.

[17] G. H. Hardy, The integration of functions of a single variable (Cambridge University Press, 2nd edition, 1928).

[18] R. Henstock, 'The efficiency of convergence factors for functions of a continuous real variable', Journal London Math. Soc. 30 (1955) 273-86, MR 17, 359.

[19] _________, 'On Ward's Perron-Stieltjes integral', Canadian Journal of Math. 9 (1957) 96-109, MR 18, 880.

[20] _________, 'Additivity and the Lebesgue limit theorems', The Greek Math. Soc. C. Carathéodory Symposium, Athens, 1973, pp. 223-41 (1974), MR 57# 6355.

[21] T. H. Hildebrandt, 'Definitions of Stieltjes integrals of the Riemann type', American Math. Monthly 45 (1938) 265-78, Zbl. 19.056.

[22] _________, 'Integration in abstract spaces', Bull. American Math. Soc. 59 (1953) 111-39, MR 14, 735.

[23] J. Kurzweil, 'Generalized ordinary differential equations and continuous dependence on a parameter', Czech. Math. Journal 7 (82) (1957) 418-46 (422-28), MR 22# 2735.

[24] H. Lebesgue, 'Intégrale, longueur, aire', Annali Mat. Pura Appl. (3) 7 (1902) 231-359, Jbuch 33, 307.

[25] _________, 'Sur les integrales singulières', Annls. Fac. Sci. Univ. Toulouse, ser. 3, 1 (1909) 25-117.

[26] _________, 'Lecons sur l'intégration et la recherche des fonctions primitives, 2nd. Ed. (1928).

[27] B. Levi, 'Teoria de la Integral de Lebesgue independiente de la nocion de medida', Publ. Inst. Mat. Univ. Nac. Litoral 3 (1941) 65-116, MR 3, 227.

[28] W. A. J. Luxemburg, 'Arzelà's dominated convergence theorem for the Riemann integral', American Math. Monthly 78 (1971) 970-79, MR 45# 6992.

[29] J. Marcinkiewicz and A. Zygmund, 'On the differentiability of functions and summability of trigonometrical series', Fundamenta Math. 26 (1936) 1-43, Zbl. 14. 111.

[30] O. Perron, 'Ueber den Integralbegriff', Sitzungsber. - Heidelberg. Akad. Wiss. Math.-Natur. Kl. 16 (1914) 1-16.

[31] S. Saks, Theory of the Integral (2nd English Edition, 1937 Zbl. 17. 300.

[32] W. Sierpinski, 'Sur un problème concernant les ensembles mesurables superficiellement', Fundamenta Math. 1 (1920) 112-15.

[33] E. C. Titchmarsh, The Theory of Functions (2nd edition, 1939) (Zbl. 5. 210 for 1st edition).

[34] L. Tonelli, 'Sulla Nozione di Integrale', Annali di Mat. (IV) 1 (1923) 105-45.

[35] W. H. Young, 'Open sets and the theory of content', Proc. London Math. Soc. (2) 2 (1905) 16-51.

[36] __________, 'On upper and lower integration', Proc. London Math. Soc. (2) 2 (1905) 52-66; Jbuch 35, 310.