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Topology for the Spaces of Denjoy Integrable Fucntions

The question of topologizing the spaces of Denjoy integrable functions was merely touched upon by several authors (see [1], [2], [4]). This paper is entirely devoted to it. The question can be stated as follows:

If $\mathcal{D}_*$ ($\mathcal{D}$) is the space of all Denjoy-Perron i.e., in the special Denjoy sense (Denjoy-Hinčin i.e., in the general Denjoy sense) integrable functions (for definitions see [6]), then what is the most appropriate topology for it and what is the dual?

We will use the following notation:

$I = [0,1]$;

$\operatorname{var}_J F$ - total variation of a function F on an interval J;

$\operatorname{osc}_J F$ - oscillation of a function F on an interval J;

C, $\mathcal{L}$ - the spaces of all real continuous, Lebesgue integrable (resp.) functions on I.

H^* - the dual space for a linear topological space H.

If x is an integrable (in any sense) function on I, then we will write X for its primitive, i. e., $X(t) = \int_0^t x(s)\,ds$.

1. As we know

(1) $$\mathcal{L} \subset \mathcal{D}_* \subset \mathcal{D}.$$

How is the question of topologizing $\mathcal{L}$ answered?

We have the following facts:

(i) L is normed via:

(2) $$\|x\| = \int_0^1 |x(t)|\, dt = \operatorname{var}_I X , \quad x \in L .$$

(ii) $L^* = L^\infty$ in the sense that if $T \in L^*$ then there is an element $y \in L^\infty$ so that

(3) $$T(x) = \int_0^1 x(t)\, y(t)\, dt \qquad \text{for } x \in L .$$

(iii) The (Lebesgue) integral

(4) $$\int_0^1 x(t)\, y(t)\, dt$$

exists for every $x \in L$ if and only if $y \in L^\infty$.

As for $\mathcal{D}_*$ (or $\mathcal{D}$), things are different: (i) doesn't work because X is not necessarily of bounded variation, and obviously the integral (3) may not exist when $x \in \mathcal{D}_*$ ($\mathcal{D}$) and $y \in L^\infty$ (just take $x \in \mathcal{D}_* \setminus L$ and $y = \operatorname{sgn} x$).

We will analyze our problem from the point of view suggested by the way (iii) is stated. What is the equivalent of (iii) for the Denjoy integrals? To answer this, we will need the following:

Definition. If $F : I \to \mathbb{R}$ is measurable and $J \subset I$ is an interval, then we define its <u>essential oscillation</u> on J as

(5) $$\operatorname{osc\,ess}_J F = \operatorname{sup\,ess}_{x \in J} F(x) - \operatorname{inf\,ess}_{x \in J} F(x).$$

Also, let the <u>essential variation</u> $\operatorname{var\,ess}_J F$ of F on J be the least upper bound of the set of all sums

(6) $$\sum_{i=1}^{n} \operatorname{osc\,ess}_{[t_i, t_{i+1}]} F ,$$

where $\{t_0, t_1, \dots, t_n\}$ is a partition of J.

F will be called <u>of bounded essential variation</u> on J if

$$\operatorname{var}_J \operatorname{ess} F < +\infty .$$

Observation. F is of bounded essential variation if and only if it is equivalent to a function of bounded variation.

Hence, if F is a distribution, then it is of bounded essential variation if and only if its distributional derivative is a measure.

We will write $\mathcal{E}$ for the space of all $y: I \to R$ which are of bounded essential variation (we identify equivalent functions; note that equivalent functions have the same distributional derivatives).

Now we turn back to our main question. As stated in [6] p. 246, if $y \in \mathcal{E}$, then the Denjoy-Perron (Denjoy-Hinčin i.e. the wide Denjoy) integral

$$(6) \qquad \int_0^1 x(t)\, y(t)\, dt$$

exists for every $x \in \mathcal{D}_*$ ($\mathcal{D}$).

The converse is also true. This was stated and shown in [7], one can also find a proof in [3], p. 55. The theorem is also mentioned in [2], but without a proof.

2. Theorem. <u>Let</u> y <u>be a measurable function on</u> I.
(i) <u>The Denjoy-Perron integral</u> (6) <u>exists for all</u> $x \in \mathcal{D}_*$ <u>if and only if</u> y <u>is of bounded essential variation</u>.
(ii) <u>The Denjoy-Hinčin integral</u> (6) <u>exists for all</u> $x \in \mathcal{D}$ <u>if and only if</u> y <u>is of bounded essential variation</u>.

Theorem 2 suggests that $\mathcal{D}_*$ (and $\mathcal{D}$) should be given a topology such that

(7) $$\mathcal{D}_*^* = \mathcal{D}^* = E \quad .$$

In [1] Alexiewicz topologizes $\mathcal{D}_*$ by giving it the norm

(8) $$\| x \| = \sup_{t \in I} \left| \int_0^t x(s)\, ds \right| = \sup_{t \in I} |X(t)| \ .$$

Then, by applying the Hahn-Banach theorem, he shows that T is a linear functional on $\mathcal{D}_*$ continuous with respect to this norm if and only if

(9) $$T(x) = \int_0^1 x(t)\, y(t)\, dt \qquad \text{(Denjoy-Perron integral)},$$

where y is some element of E (note that equivalent functions yield the same functionals).

One can easily check that the same argument works for the Denjoy-Hinčin integral.

There is another idea about the topology for $\mathcal{D}_*$ in [2]. An element $x \in \mathcal{D}_*$ is given the norm

(10) $$|\!|\!| x |\!|\!| = \operatorname*{osc}_I X \ .$$

But $\mathcal{D}_*$, as a normed space, with $|\!|\!| \ |\!|\!|$ is isomorphic to $\mathcal{D}_*$ equipped with $\| \ \|$. In fact, for $x \in \mathcal{D}_*$ we have $X(0) = 0$, so that

(11) $$\sup_{t \in I} |X(t)| = \max \left(\sup_{t \in I} X(t), - \inf_{t \in I} X(t) \right)$$

and hence

(12) $\| x \| \leq ||| x ||| \leq 2 \| x \|$.

Thus $\| \ \|$ and $||| \ |||$ yield isomorphic normed spaces.

Now we have a (metric) topology for $\mathcal{D}_*$ (and $\mathcal{D}$) giving the desired dual. It is not complete. Its restriction to L is much weaker than the original topology of L , it is even weaker than the weak topology of L (it had better be, since it yields a smaller dual!). But it is the best possible one, i.e., it is the strongest topology generating $\mathcal{E}$ as the dual. Why?

For a linear topological space the strongest topology yielding a given dual is called the Mackey topology (see [5]). For $\mathcal{D}_*$ (and $\mathcal{D}$) with $\mathcal{E}$ as the dual, our topology turns out to be Mackey. This is a consequence of the following:

3. Theorem. If V is a locally convex pseudo-metrizable space, then the topology of V is Mackey.

For the proof see [5] p. 210.

Now for a couple of words about the topology on $\mathcal{E}$. If we define

$$A = \left\{ X \in C \mid X \text{ is a primitive of an } x \in \mathcal{D}_* \right\},$$

(12)

$$B = \left\{ X \in C \mid X \text{ is a primitive of an } x \in \mathcal{D} \right\},$$

and consider A and B as subspaces of

$$C_0 = \left\{ X \in C \mid X(0) = 0 \right\},$$

then we have isometries

(13) $\mathcal{D}_* \rightarrow A \quad , \quad \mathcal{D} \rightarrow B$

given by $x \longmapsto X$.

Both $\mathcal{A}$ and $\mathcal{B}$ are dense in C_0, thus their duals are identical with $C_0^* = \mathcal{M}$ (Borel measures on I). Therefore the appropriate topology is the one given by the norm

(14) $\| y \| = |Dy| (I), \quad y \in E$,

where Dy is the measure generated by y, i.e., its distributional derivative.

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