Andrew Bruckner, Department of Mathematics, University of California, Santa Barbara, CA 93106

Some New Simple Proofs of Old Difficult Theorems

1. Introduction.

In his doctoral dissertation published in 1799, Gauss gave the first formal proof of what we now call the Fundamental Theorem of Algebra. The result had been conjectured much earlier by Girard and various 18th century mathematicians had spent considerable effort in attempts to prove it. In fact, D'Alembert put such an effort into settling it that the theorem is still widely known in France as D'Alembert's Theorem. (D'Alembert actually provided a proof in 1743, but it wouldn't meet today's standards of rigor). Gauss actually published several proofs of this theorem (one as late as 1850, when he was in his seventies) in an attempt to find one which is entirely algebraic. He didn't succeed. In fact, the first algebraic proof may be due to Perron in 1951! (See Zassenhaus [28]).

Today, most students of mathematics learn proofs based on Liouville's theorem, Rouche's theorem, or on various other standard theorems in analytic function theory. Because of the simplicity of these proofs, many students have no idea of the difficulties encountered by Gauss and his predecessors in proving the theorem.

The foregoing provides a celebrated instance of a situation that often occurs in mathematics. A problem is solved with great difficulty and much ingenuity and after a long period of time. Later, when more machinery is available, the solution flows easily from the new machinery, which may have been created with entirely different objectives in mind.

The purpose of this article is to discuss two related problems of this type in differentation theory. We also discuss a third such problem which the new machinery failed to solve. Instead, it succeeded in indicating "why" the problem was so difficult. This in itself was no easy task at the time.
We show it could have been!

Specialists in differentiation theory will be familiar with all these problems, though not necessarily with their histories or with all the new proofs.

2. Differentiable nowhere monotonic functions.

Slightly over one hundred years ago Du Bois-Reymond expressed the view that a nowhere monotonic function cannot be differentiable. Dini, on the other hand, believed the existence of such functions highly probable [12 : p. 412].

In 1887 Köpcke provided a construction of such a function. In discussing Köpcke's work, Denjoy wrote in 1915 [8 : p. 228] "In 1887, Köpcke gave in Math-Annalen an example of a function possessing in each point (or so he thought) a derivative which vanished and took both signs in every interval contained in its domain of definition. This geometor returned to this subject on several occasions [references], correcting each time the errors contained in the previous proofs. This question of differentiable, nowhere monotonic functions has also provoked many other works [references]."

The various constructions Denjoy referred to were quite complicated. At this point in Denjoy's paper [8], he had already given three separate constructions. He was about to give a Köpcke-type construction, but before doing so he alerted the reader to the "clarity and simplicity" of his construction made possible by borrowing ideas from "Borel and Lebesque on the measure of sets." To Denjoy, his constructions were simple and clear. We would probably find them horrendous.

Hobson modifies Pereno's modification of Köpchke's construction in his book [12]. This was published about forty years after Köpchke's first correction, thirty years after Pereno's modification and fifteen years after Denjoy's "simple and clear" development. It required ten pages!

Today a number of proofs are available. Some are constructive in nature - Zahorski's well-known construction [26], and recent relatively elementary constructions by Katznelson and Stromberg [17] and by Blazek, Borak and Maly [3]. The others depend on machinery not available to Hopson or his predecessors. We discuss briefly several such proofs. We mention that some of the proofs are based on theorems which, directly or indirectly, depend on results obtained by Zahorski in [26].

a) Baire Category Theorem [25]. Weil's proof of the existence of differentiable nowhere monotonic functions has a special appeal. It does not rely on difficult constructions; nor does it depend on other theorems which were difficult to prove and whose proofs depend on other theorems, which were also difficult to prove. In short, it can be presented to students who have seen the Baire Category Theorem, know that the uniform limits of derivatives are derivatives, and who believe the Pompeiu construction of a strictly increasing differentiable function whose derivative vanishes on a dense set. Of all the known proofs, it is the one which can be read and understood with confidence of the correctness of the result more easily than any other.

Let Δ' be the Banach space of bounded derivatives on $[0,1]$ furnished with the sup norm. Let $\Delta_0' = \{f \in \Delta' : f = 0$ on a dense set$\}$. It is easy to verify that Δ_0' is closed under addition and is complete. Straightforward use of Pompeiu derivatives establishes that for each interval $I \subset [0,1]$, $\{f \in \Delta_0' : f \geq 0$ on $I\}$ is nowhere dense in Δ_0'. The result follows readily from the Baire Category Theorem.

Here is a plausible approach to constructing a nowhere monotonic differentiable function. First construct two derivatives g_1 and g_2 in Δ_0^1 such that the sets

$$\{x : g_1(x) > 0 \text{ and } g_2(x) = 0\}$$

and $$\{x : g_2(x) > 0 \text{ and } g_1(x) = 0\}$$ are dense.

Let $g = g_1 - g_2$. Then $g \in \Delta_0^1$. The function

$G(x) = \int_0^x g(t)dt$ has the desired properties.

To construct such a pair (g_1, g_2) "from scratch" is not an easy task. Recent works in differentiation theory have provided theorems which permit almost obvious uses of this approach.

b) Density Topology. Goffman [11] made clever use of properties of the density topology to construct such a pair (g_1, g_2).

This topology is completely regular, countable sets are closed, and the continuous functions (with density topology as domain and euclidean topology as range) are the approximately continuous functions. Let A and B be disjoint countable dense subsets of R : $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$. For each $n = 1,2,\ldots,$ let f_n be approximately continuous on R, $0 \le f_n \le 1$, $f_n(a_n) = 1$, $f = 0$ on B. Choose g_n approximately continuous on R_1 $0 \le g_n \le 1$, $g_n(b_n) = 1$ and $g = 0$ on A. Since the uniform sum of a series of approximately continuous functions is also approximately continuous, the function

$$g = \sum_{n=1}^{\infty} \frac{f_n}{2^n} - \sum_{n=1}^{\infty} \frac{g_n}{2^n}$$

is a bounded approximately continuous function. It clearly has the desired properties.

c) Extensions to derivatives [21]. Petruska and Laczkovich proved that if Z is a null set contained in $[0,1]$, then the restriction of any Baire 1 function to Z can be extended to a derivative on $[0,1]$.

Let

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, \ q \text{ even} \\ -\frac{1}{q} & \text{if } x = \frac{p}{q}, \ q \text{ odd} \\ 0 & \text{elsewhere} \end{cases}$$

with p and q relatively prime. Then f is Baire 1 and its restriction to the rationals in $[0,1]$ can be extended to a derivative $\hat{f}$ on $[0,1]$. Any primative of $\hat{f}$ is nowhere monotonic.

d) <u>Products of derivatives</u>. Mařík and Weil [7] have proved, (with only minor assistance from me), that every Baire 1 function that vanishes almost everywhere is a product of two derivatives. Let f be the function of example c). Write $f = gh$, $(g, h \in \Delta')$.

Let $A = \{x : g(x) > 0, \ h(x) > 0\}$

$B = \{x : g(x) > 0, \ h(x) < 0\}$

$C = \{x : g(x) < 0, \ h(x) > 0\}$

and $D = \{x : g(x) < 0, \ h(x) < 0\}$

Since f is positive on a dense set, the set $A \cup D$ is dense. Thus there exists an interval $I \subset (0,1)$ on which A

is dense or D is dense. Similar reasoning leads to an interval $J \subset I$ on which B is dense or C is dense. On J, we have both members of one of the pairs (A,B), (A,C), (D,B) and (D,C) dense. Each of these cases leads to one of the derivatives g or h taking both signs on J. A primitive of this function has the desired properties on J.

Another approach has surfaced in recent years. One first obtains a function with the desired oscillatory behavior. This function need not be differentiable. One then transforms this function via a transformation which creates differentiability without destroying the desired oscillatary behavior. One can use a similar approach to create derivatives which take both signs in every interval. The three examples following use one of these approaches.

e) <u>Change of Variable, and changes of scale</u>.

(i) Let $E \subset [0,1]$ be measurable and together with its complement have positive measure in every subinterval of $[0,1]$

$$\text{Let } f(x) = \begin{cases} 1 & \text{if } x \in E \\ -1 & \text{if } x \notin E \end{cases} \quad \text{and let } F(x) = \int_0^x f(t)dt.$$

Then F is absolutely continuous. Fleissner and Foran [9], Bruckner and Goffman [6], and Kaplan and Slobodnick [15] have all established theorems which show that there exists a homeomorphism h of $[0,1]$ onto itself such that $F \circ h$ is differentiable. It is clear that F is nowhere monotonic so the same is true of the differentiable function $F \circ h$.

(ii) A similar proof can be based on another theorem of Fleissner and Foran [10]. This theorem guarantees the existence of a homeomorphism H of R onto R such that $H \circ F$ is differentiable (F as above). Again, it is clear that $H \circ F$ is nowhere monotonic

These proofs were articulated in Kaplan and Slobodnick [15]

(iii) The two preceding proofs involved changes of variables or of scale that created differentiable functions with the desired properties. One can also call for a proof which creates derivatives having the desired properties without introducing the primitives. Such a proof is easy to devise using the Maximoff-Preiss Theorem [22] which asserts that every Darboux-Baire 1 function can be transformed into a derivative via a homomorphic change of scale.

Let $\{A_n\}_0^\infty$ be a sequence of pairwise disjoint, perfect subsets of $[0,1]$ such that $\bigcup_{n=0}^{\infty} A_{2n}$ has positive measure in every subinterval of $[0,1]$ and $\bigcup_{n=0}^{\infty} A_{2n+1}$ has positive measure in every subinterval of $[0,1]$. Define function f on $[0,1]$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in A_{2n+1} \\ -\frac{1}{n} & \text{if } x \in A_{2n} \\ 0 & \text{elsewhere} \end{cases}$$

It is easy to verify that f is a Baire 1 function. There exists [5] a Darboux-Baire 1 function g such that $g = f$ a.e. In particular, g takes both signs in every interval. An application of the Maximoff-Preiss theorem gives the desired derivative.

It may be of interest to note that no comparable proof based on outside compositions with homeomorphisms is readily available. This is so because the class of functions of the form $H \circ f$ (H a homeomorphism of R onto R, $f \in \Delta'$) has not been characterized. It is clearly a proper subset of the first Baire class (in fact, it is clearly contained in Zahorski's class $\mathcal{M}_4$), but we know of no simple-to-construct function f such that $H \circ f$ has the desired properties for some H.

We turn now to a related problem. In 1882 Hankel (see [12]) tried to construct a differentiable function with dense sets of strict maxima and strict minima. He did not succeed. (A solution to this problem would automatically provide a solution to the previous problem as well). Somewhat later Zalcwasser [27] solved a more difficult problem: given two countable, disjoint sets A and B, does there exist a differentiable function with strict maxima on A, strict minima on B, and no other extrema? He answered this question affirmatively in his lengthy paper [27]. Recently Kelar [16] provided a simpler proof. He constructed a Lipschitz function with the desired extrema properties. An application of the Fleissner-Foran theorem noted in c ii) above completes the proof.

A possibly simpler proof can take the following form. (Here we assume A and B are both dense - Zalcwasser showed the general case reduces to this one readily). It is not difficult to show [4] that if A and B are disjoint, dense, and denumerable subsets of [0,1], and A' and B' have the same properties, then there exists a homeomorphism g of [0,1] onto itself such that $g(A) = A'$, $g(B) = B'$ and $1/2 \le \frac{g(y) - g(x)}{y - x} \le 2$. for all $x \neq y$ in [0,1]. It is easy to verify that the set of derivatives in Weil's class Δ_0' which have primitives all of whose local extrema are strict, is rendual in Δ_0'. Let F be a primitive of such a derivative and let A' and B' be its set of strict maxima and strict minima respectively. Let g be the homeomorphism described above. Then $F \circ g$ is a Lipschitz function and has the desired extrema properties. The Fleissner - Foran theorem produces a homeomorphism of R onto R such that $h \circ (F \circ g)$ is differentiable. This function has all required properties.

Remark: One can actually take g differentiable, so $F \circ g$ has the desired properties. We can thus assert that the typical differentiable nowhere monotonic Lipschitz function F with $F(0) = 0$ (relative to the norm $\|F\| = \sup_{t \in [0,1]} |F'(t)|$) has the property that given A and B denumerable, dense, disjoint, there exists a diffeomorphism G such that $F \circ G$ achieves strict local maxima on A, strict local minima on B, and no other local extrema.

The previous problems dealt with the pathological behavior of differentiable functions. We turn now to a discussion of a question cencerning pathology with respect to differentiative of continuous functions.

The fact that there exist continuous nowhere differentiable functions has been known for over a century. The early proofs by Weierstrass and others were by example. It wasn't until 1931 that Banach [1] and Mazurkiewicz [18] gave existence proofs via the Baire Category Theorem.

Weierstrass' example W has the property that unilateral derivatives W'_+ and W'_- exist, but are infinite, on dense sets. The question arose whether there exists a continuous function f such that f'_+ and f'_- exist nowhere (finite or infinite). It took many years before Besicovitch [2] gave an example of a continuous function B such that $D_+B < D^+B$ and $D_-B < D^-B$ everywhere. His example was simplified by Pepper [20], a modified version appearing in Jefferey [14]. Later, Morse [19] provided another example based on arithmetic rather than geometric considerations. Needless to say, all these examples are very complicated - even Jefferey's required almost ten pages of his book. And in reviewing earlier works on the subject, Morse points out that some doubt still remained about the existence of such functions.

While visiting Harward, Saks [23] approached the problem via the Baire Category Theorem. He observed that straightforward proofs indicated that continuous functions typically have finite or infinite derivatives or unilateral finite derivatives nowhere. He then proved, however, that they do have unilateral infinite derivatives on sets of cardinality of the continuum. (By a "typical" property, we mean, as usual, that the functions exhibiting that property form a residual subset of C[0,1]). Saks' proof was far from elementary. It involved showing that such functions form a subset of C[a,b] which is second category in every sphere and which is analytic. This implies that the class of such functions is residual in C[a,b]. Saks made use of some results that Tarski and Kuratowski had recently obtained.

Thus Saks did not find a simple proof of the existence of Besicovitch-type functions. Some mathematicians view his result as showing why examples of such functions may be so difficult to provide. They are atypical (Undergraduates may have difficulty accepting this viewpoint - they have no difficulty providing examples of integers or of differentiable functions).

A very simple proof of Saks' result was communicated by Laczkovich who attributed it to Preiss. Oddly, this proof uses only results which were available in 1934 - not at the time Saks wrote his paper [23], but surely at the time he wrote his book [24].

Lemma. Every continuous function F has a unilateral approximate derivative (finite or infinite) on some set having the cardinality of the continuum.

Proof: Let

$$A = \{x : D^+F = -\infty\}$$

$$B = \{x : -\infty < D^+F < 0\}$$

$$C = \{x : 0 \leq D^+F \leq \infty\}$$

If $A \neq \phi$, then for every $x \in A$, $F'_{+ap}(x) = -\infty$. Thus, if A has cardinality of the continuum, it provides the required set. On B, F is VBG and therefore approximately differentiable a.e. Thus, if B has positive measure, it provides the required set. Finally, if the cardinality of A is less than that of the continuum (so A is denumerable since it is a Borel set) and B has measure 0, then F is nondecreasing by a standard theorem. In that case F is differentiable almost everywhere.

Theorem: (Saks) The typical continuous function F has infinite unilateral derivatives on sets of cardnality of the continuum.

Proof. (Laczkovich attributed to Preiss)

The typical continuous function is nowhere approximately differentiable [13]. Thus, with A, B and C as in the lemma, B has measure zero. If A has cardinality less than that of the continuum, then (as in the proof of the lemma) F is nondecreasing. It isn't!

Thus, the <u>typical</u> continuous function has infinite unilateral derivatives over sets of cardinality of the contimium; but <u>every</u> continuous function has unilateral approximate derivatives on such sets.

The foregoing provides a very simple proof of a difficult theorem of Saks. But it does not provide a simple proof of the existence of Besicovitch-type functions. Does there exist a complete matric ρ on the continuous functions with respect to which the Besicovitch-type functions are residual?

REFERENCES

1. S. Banach, Über die Bairische Kategorie gewisser Funktionenmengen, Studia Math. 3 (1931), 174-197.

2. A. Besicovitch, Diskussion der stetigen Functionen in Zusammenhang mit der Frage über ihre Differentierbarkeit, Bull. Acad. de Russie 19 (1925), 527-540.

3. J. Blažek, E. Borak, J. Maly, Čas. pro. pẽst mat. roč, 103 (1978), 53-61.

4. A. Bruckner, Some indirect consequences of a theorem of Zahorski, to appear.

5. A. Bruckner, J. Ceder, and R. Keston, Representations and approximations by Darboux functions in the first class of Baire, Rev. Roum. Math Pures et Appl. 13 (1968), 1247-1254.

6. A. Bruckner and C. Goffman, Differentiability through change of variables, Proc. Amer. Math. Soc 61 (1976), 235-241.

7. A. Bruckner, J. Mařík and C. Weil, Baire one null functions, to appear.

8. A. Denjoy, Sur les fonctions dérivées sommables, Bull. Soc. Math. France, 43(1915), 161-248.

9. R. Fleissner and J. Foran, A note on differentiable functions, Proc. Amer. Math. Soc. 69 (1978), 56.

10. R. Fleissner and J. Foran, Transformations of differentiable functins, Colloq. Math. 39 (1976) 278-281.

11. C. Goffman, Everywhere differentiable functions and the density topology, Proc. Amer. Math. Soc. 51 (1975) 250.

12. E. Hobson, Theory of functions of a real varialbe, vol. 2. Dover, New York 1957.

13. V. Jarník, Sur la dérivabilité des functions continues, Spisy Privodau, Fak. Univ. Karlovy, 129 (1934) 3-9.

14. R. Jeffery, The theory of functions of a real variable, Mathematical Expositions No. 6, University of Toronto Press, 1962.

15. L.I. Kaplan and S.G. Slobodnick, Monotone Transformations and differential properties of functions, Mat. Zanetki 22 (1977), 859-871.

16. V. Kelar, On strict local extrema of differentiable functions, Real Anal. Exch. 6 (1980-1981), 242-244.

17. Y. Katznelson and K. Stromberg, Amer. Math. Monthly, 81 (1974), 349-354.

18. S. Mazurkiewicz, Sur les functions non-derivables, Studia Math 3 (1931), 92-94.

19. A. Morse, A continuous function with no unilateral derivatives, Trans. Amer. Math. Soc. 44 (1938), 496-507.

20. E. Pepper, On continuous functions without a derivative, Fund. Math 12 (1928) 244-253.

21. G. Petruska and M. Laczkovich, a theorem on approximately continuous functions, Acta Math. Acad. Sci. Hung. 24 (1974) 383-387.

22. D. Preiss, Maximoff's Theorem, Real Anal. Exch. 5 (1979-1980), 92-104.

23. S. Saks, On the functions of Besicovich in the space of continuous functions, Fund. Math. 19 (1932), 211-219.

24. S. Saks, Theory of the integral, Monografie Matematyczne 7, Warszawa-Lwów, 1937.

25. C. Weil, On nowhere monotone functions, Proc. Amer. Math. Soc. 56 (1976) 388-389.

26. Z. Zahorski, Sur la premiére derivée, Trans. Amer. Math. Soc. 69 (1950) 1-54.

27. Z. Zalcwasser, O funkcjach Köpckego, Prace Math. Fiz 35 (1927-1928) 57-99 (Polish; French Summary).

28. H. Zassenhaus, On the Fundamental Theorem of Algebra, Amer. Math. Monthly, 74 (1967) 485-497.