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THE SETS OF CONSTANCY OF FUNCTIONS WITH A VANISHING DERIVATIVE

In [3] H. Whitney raised the following problem: How far should a planar curve α be from rectifiable in order that it support a nonconstant real function $f:\alpha\to\mathbb{R}$ such that for every $z_0\varepsilon\alpha$,

$$|f(z_0)-f(z)| = o(|z_0-z|) \quad (z\to z_0,\ z\varepsilon\alpha)\ ?$$

In his paper a construction is given of an example of α and an $f:\alpha\to\mathbb{R}$ with the above property, and in general we call a set $H\subset\mathbb{R}$ a W(hitney) set if there exists a nonconstant $f:\alpha\to\mathbb{R}$ with a vanishing derivative. A set of <u>constancy</u> (C-set) is a non-W-set. To give a complete characterization of W-sets is an open problem and it does not seem to be easy by any means. A deep positive result was proved by Choquet in 1944, [2]: if the planar curve α is the graph of a real function $f:[0,1]\to\mathbb{R}$, then α is a C-set. Applying the basic idea of Choquet's proof we were able to prove the following criterion for a set to be a C-set.

<u>If the simple arc</u> α <u>has a parameterization</u> $\psi:[a,b]\to\alpha$ <u>such that the set of "points of expansion"</u>

$$A = \left\{z_0\varepsilon\alpha:\ \frac{|\psi(t)-\psi(t_0)|}{|t-t_0|}\to\infty,\ t\to t_0+0,\ t_0=\psi^{-1}(z_0)\right\}$$

<u>is</u> σ-<u>finite with respect to one dimensional Hausdorff measure</u>, λ, <u>then</u> α <u>is a</u> C-<u>set</u>.

It is not difficult to verify directly that, for an ordinary graph, the set A is always λ-σ-finite and hence our result is an extension of Choquet's theorem. By some results of Besicovitch and Schoenberg [1] an ordinary graph can have Hausdorff dimension 2, and hence large Hausdorff dimension by itself cannot imply the W-property. In the reverse direction, we proved that if the set A is the image of a set of positive measure, then α is a W-set. All these show, contrary to Whitney's original assumption, that it is not a global property (like rectifiability or nonrectifiability) of the entire curve which determines the W-property, but rather it is the size of the set of points of expansion which determines whether a curve is a W-set or C-set.

REFERENCES

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[3] H. Whitney, A function not constant on a connected set of critical points, Duke Math. J. 1 (1935) 514-517.