Richard O'Malley, Department of Mathematical Sciences(*)
University of Wisconsin-Milwaukee, Milwaukee, WI 53201

THE MULTIPLE INTERSECTION PROPERTY FOR PATH DERIVATIVES

The purpose of the multiple intersection property is to extend to path derivatives a result previously established for approximate derivatives [2]. That result is the decomposition property of approximate derivatives. More precisely we deal with two ideas: see [1], [2], [3].

(1) A function $f: R \to R$ is said to be compositely differentiable to another function $g: R \to R$ if there is a sequence of closed sets X_n such that $\bigcup_{n=1}^{\infty} X_n = R$ and for each n, the restriction of f to X_n differentiates to the restriction of g to X_n. (The functions $g|X_n$ are called a decomposition of g.)

(2) A collection of sets, $E = \{E_x, x \varepsilon R\}$ is called a path system if, for each x, E_x contains x and has x as a limit point. Then, relative to this system, $f: R \to R$ is said to have $g: R \to R$ as path derivative, if, for each x, the restriction of f to E_x, differentiates, at x, to g(x).

In general (1) is a more restrictive condition than (2). However, it should be noted that (2) requires that g(x) is a derived number of f at x while (1) does not. That is, some of the X_n may have isolated points. But we have the following:

Proposition 1: If $f: R \to R$ is compositely differentiable to $g: R \to R$ and, for each x, g(x) is a derived number of f at x, then there is a sequence of perfect sets P_n such that $\bigcup_{n=1}^{\infty} P_n = R$ and f is compositely differentiable to g relative to P_n.

Thus (1) implies (2) for such f and g.

It should be clear that in general (2) does not imply (1). In [1] it was found that many path derivates will have properties of derivatives if some kind of intersection condition is placed on the path system. A similar situation exists here.

(3) A system of paths, $E = \{E_x, x \in R\}$, has the multiple intersection property if there is a positive function δ, defined on R, such that, for each triple of numbers $x_1 \leq x_2 \leq x_3$ not all equal, if

$$[x_1, x_3] \subset \bigcap_{i=1}^{3} (x_i - \delta(x_i),\ x_i + \delta(x_i)), \text{ then}$$

$$\left(\bigcap_{i=1}^{3} E_{x_i}\right) \cap [x_j, x_{j+1}] \neq \phi \quad \text{for } j = 1,2.$$

In the case where $x_j = x_{j+1}$ for j = 1 or 2 this condition reduces to the intersection property of [1]. Then we have:

Theorem 1. If $f: R \to R$ has $g: R \to R$ as a path derivative relative to a system, $E = \{E_x, x \in R\}$ where E has the multiple intersection property, then f is compositely differentiable to g.

Further we have, using Proposition 1,

Theorem 2. If $f: R \to R$ is compositely differentiable to $g: R \to R$ and, for each x, g(x) is a derived number of f at x, then there is a path system E having the multiple intersection property such that g is the path derivative of f relative to E.

References

[1] A. Bruckner, R. O'Malley, B. Thomson, Path derivatives: A unified view of certain generalized derivatives, to appear in Transactions of the A.M.S.

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[3] R. O'Malley, C. E. Weil, Selective, Bi-selective, and Composite Differentiation to appear in Acta Math. Acad. Sci. Hung.