Baire 1 functions(1)

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1. History. The transfinite classification of real functions was introduced by René Baire in his thesis [1] in 1899. Baire's starting point was the following problem.

Let f be a real function of two variables and suppose that f is separately continuous in each of its variables. How can we characterize the diagonal function $f(x,x)$? (One can show by simple examples that it is not necessarily continuous.) The answer is given by the following theorem proved in the first part of Baire's thesis.

Theorem. For any real function $\varphi:\mathbb{R} \to \mathbb{R}$ the following assertions are equivalent:

(i) φ is the limit of a convergent sequence of continuous functions.

(ii) For every non-empty, perfect $P \subset \mathbb{R}$, the restriction of φ to P has a point of continuity.

(iii) There exists a separately continuous function f of two variables such that

$$f(x,x) = \varphi(x) \qquad (x \in \mathbb{R}).$$

Then Baire introduced the first class, B_1, as the collection of those functions which are not continuous but possess property (i). He also defined B_α for every countable ordinal α:

(1) Results that are not referenced have not been published.

if B_β have been defined for every $\beta < \alpha$ then

$$B_\alpha = \{f = \lim_{n\to\infty} f_n \; ; \; f_n \in \bigcup_{\beta<\alpha} B_\beta \quad (n = 1,2,\dots)\} \setminus \bigcup_{\beta<\alpha} B_\beta \,.$$

(Later it turned out to be more convenient not to exclude the functions belonging to the previous classes; so that the "Baire classes" today are not the same as those defined by Baire himself.) Baire remarks that the class

$$E = \bigcup_{\alpha<\omega_1} B_\alpha$$

is closed under pointwise convergence and is of the power of the continuum. Hence E does not contain every function. Thus the question arises, what properties characterize the functions in E. He introduced the "Baire property" as an attempt to find such a characterization. It has to be mentioned that the notion of category of sets and the category theorem also originate from this paper.

Lebesgue proved as early as in 1899, that the properties (i) and (ii) are also equivalent for functions of n variables [12]. (The proof given by Baire used the special structure of perfect sets in $\mathbb{R}$ and could therefore not be generalized. Later Baire gave another proof [4].) In 1904 Lebesgue proved the following theorem (see [13] or [5], p. 154). Let f be defined on an interval $I \subset \mathbb{R}^n$. Then f is of the first class if and only if for every $\epsilon > 0$ there is a sequence

$\{F_n\}_{n=1}^{\infty}$ of closed sets such that $\bigcup_{n=1}^{\infty} F_n = I$ and $\omega(f,F_n) < \epsilon$ $(n = 1,2,\ldots)$, where $\omega(f,H)$ denotes the oscillation of f on the set H. Lebesgue remarks that this condition is equivalent to the following one:

For every $a < b$, the associated set $\{x \in I;\ a < f(x) < b\}$ is F_σ.

In his paper [14] Lebesgue completely solved the general problem posed by Baire. In this monumental paper Lebesgue introduced the transfinite classification of Borel sets and proved that $f \in \bigcup_{\beta \leq \alpha} B_\beta$ if and only if the associated set $\{x;\ a < f(x) < b\}$ is of additive class α for every $a < b$. He showed that property (ii) has a generalization for every $\alpha < \omega_1$, proved that the classes B_α are non-empty and that the Baire property is valid for every $f \in E$. (Baire himself only proved this for $f \in B_1 \cup B_2$.) It is worth mentioning that this very same paper contains the famous erroneous assertion that the projections of the Borel subsets of the plane are also Borel.

Let f be defined on an interval $I \subset \mathbb{R}^n$. By the theorems of Baire and Lebesgue, the following conditions are equivalent:

(A) $\{x\ ;\ f(x) < c\}$ and $\{x\ ;\ f(x) > c\}$ are F_σ sets for every real c.

(B) f is the limit of a pointwise convergent sequence of continuous functions.

(C) For every non-empty, perfect subset P of the domain of f, $f|_P$ has a point of continuity.

It turned out soon that these results can be generalized to functions defined on some other spaces. The first step in this direction was taken by Baire himself. As early as 1899, Baire introduced the space ω^ω and defined the notions of limit point, closed set, nowhere dense set and sets of first and second category in ω^ω. He also proved the category theorem in ω^ω [2], [3].

The general notion of metric space was introduced by Fréchet in 1906 [6]. Shortly thereafter it was proved that (A) and (B) are equivalent in every metric space (see [10], p. 248) and that (C) implies (B) if the space is separable ([10], p. 254). If the space is complete, then (B) implies (C) ([10], p. 253). In general this is not true: in the space of rational numbers (A) and (B) hold for every function while there are nowhere continuous functions which obviously do not have property (C).

Finally Montgomery proved in 1935 that (C) implies (B) in <u>every</u> metric space [15]. Hence (A) $\Leftrightarrow$ (B) $\Leftarrow$ (C) is always true and (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C) is true if the space is complete.

2. Baire 1 functions and associated sets.

In an arbitrary topological space the conditions (A) and (B) are no longer equivalent. Consider the set $\mathbb{R}$ of real numbers endowed with the density topology. It is well-known that the continuous functions in this topology are the approximately continuous functions and that they are Baire 1 functions in the ordinary topology. It follows that any function satisfying condition (B) in the density topology is a Baire 2 function. (By a theorem of Preiss [16] the converse is also true.) On the other hand, it is easy to check that the F_σ sets in the density topology are exactly the Lebesgue measurable sets and hence the condition (A) is equivalent to the measurability of the function in question. Therefore, in the density topology, property (B) is strictly stronger than property (A).

Let $(X,\mathcal{T})$ be a topological space. We shall denote by $\mathcal{A}_1(X)$ and $\mathcal{B}_1(X)$ the classes of functions $f:X \to \mathbb{R}$ satisfying the conditions (A) and (B), respectively.

$$\mathcal{A}_1(X) = \{f:X \to \mathbb{R} \;;\; \{x \in X \;;\; f(x) > c\} \text{ and } \{x \in X \;;\; f(x) < c\} \text{ are } F_\sigma \text{ sets in } X \text{ for every } c \in \mathbb{R}\},$$

$$\mathcal{B}_1(X) = \{f:X \to \mathbb{R} \;;\; f = \lim_{n\to\infty} f_n,\ f_n \in C(X)\ (n=1,2,\dots)\},$$

where $C(X)$ denotes the set of real valued, continuous functions defined on X.

(Since, in general, $\mathcal{A}_1(X)$ and $\mathcal{B}_1(X)$ do not coincide, we have to decide which one is to be called the first class of Baire in X. $\mathcal{B}_1(X)$ seems to be the natural choice.)

Let $\mathcal{Z}$ denote the system of zero-sets in X:

$$\mathcal{Z} = \{f^{-1}(\{0\}) \;;\; f \in C(X)\}$$

and let

$$\mathcal{Z}^\sigma = \{ \bigcup_{n=1}^{\infty} H_n \; ; \; H_n \in \mathcal{Z} \; (n=1,2,\ldots) \}.$$

<u>Proposition 2.1</u>. $f \in \mathcal{B}_1(X)$ if and only if $\{x \in X \; ; \; f(x) < c\} \in \mathcal{Z}^\sigma$ and $\{x \in X \; ; \; f(x) > c\} \in \mathcal{Z}^\sigma$ for every $c \in \mathbb{R}$.

(This is a special case of some general theorems on pointwise limits of functions belonging to a given class; see [10], p. 241 or [9], p. 248.)

<u>Corollary 2.2</u>. $\mathcal{B}_1(X) \subset \mathcal{A}_1(X)$ holds in every topological space $(X, \mathcal{T})$.

<u>Corollary 2.3</u>. If $\mathcal{Z}^\sigma$ contains the closed sets then $\mathcal{A}_1(X) = \mathcal{B}_1(X)$.

<u>Theorem 2.4</u>. If $(X, \mathcal{T})$ is normal then $\mathcal{A}_1(X) = \mathcal{B}_1(X)$.

(We remark that normality does not imply that $\mathcal{Z}^\sigma$ contains the closed sets.)

Since the density topology is completely regular [8], the condition of normality cannot be replaced by complete regularity.

<u>3. Baire 1 functions and continuity points</u>.

Let $(X, \mathcal{T})$ be a topological space. We introduce the following classes of real valued functions defined on X.

$\mathcal{C}_1(X) = \{f:X \to \mathbb{R}$; for every non-empty, closed $Y \subset X$, the restriction of f to the subspace Y has a continuity point$\}$,

$\mathcal{D}(X) = \{f:X \to \mathbb{R}$; for every $Y \subset X$, the set of points of discontinuity of the restriction of f to the subspace Y is of first category in $Y\}$,

$\mathcal{B}_1^*(X) = \{f:X \to \mathbb{R}$; for every non-empty $Y \subset X$ and $\epsilon > 0$ there exists an open $G \subset X$ such that $Y \cap G \neq \emptyset$ and $\omega(f, Y \cap G) < \epsilon\}$.

<u>Proposition 3.1</u>. In every topological space $(X, \mathcal{T})$,
$\mathcal{C}_1(X) \subset \mathcal{B}_1^*(X) \subset \mathcal{D}(X)$ and
$\mathcal{B}_1(X) \subset \mathcal{A}_1(X) \subset \mathcal{D}(X)$.

(The first assertion is an easy consequence of the definitions. As for $\mathcal{A}_1(X) \subset \mathcal{D}(X)$, see [11], p. 394.)

A topological space is said to be a Baire-space if it does not contain any non-empty open set of first category. This definition together with the previous assertion immediately implies the following:

<u>Proposition 3.2</u>. If the closed subspaces of X are Baire-spaces then

$$\mathcal{B}_1(X) \subset \mathcal{A}_1(X) \subset \mathcal{B}_1^*(X) = \mathcal{C}_1(X) = \mathcal{D}(X).$$

<u>Corollary 3.3</u>. If $(X, \mathcal{T})$ is a compact Hausdorff space then

$$\mathcal{A}_1(X) = \mathcal{B}_1(X) \subset \mathcal{B}_1^*(X) = \mathcal{C}_1(X) = \mathcal{D}(X).$$

Theorem 3.4. In every topological space $(X,\mathcal{T})$, the following are equivalent:

(i) $\mathcal{C}_1(X) \subset \mathcal{A}_1(X)$,

(ii) $\mathcal{B}_1^*(X) \subset \mathcal{A}_1(X)$,

(iii) the space is perfect and for every $Y \subset X$, if Y is locally F_σ then Y is F_σ. (Y is said to be locally F_σ if every $x \in Y$ has an open neighborhood U such that $Y \cap U$ is F_σ.)

Corollary 3.5. If $(X,\mathcal{T})$ is perfect and paracompact then

$$\mathcal{C}_1(X) \subset \mathcal{B}_1^*(X) \subset \mathcal{B}_1(X) = \mathcal{A}_1(X) \subset \mathcal{D}(X).$$

(Observe that every metric space is perfect and paracompact.)

Proposition 3.6. In every topological space, if $\mathcal{C}_1(X) \subset \mathcal{B}_1(X)$ then $\mathcal{A}_1(X) = \mathcal{B}_1(X)$.
(Proof: Let $F \subset X$ be closed and let f be the characteristic function of F. Then $f \in \mathcal{C}_1(X) \subset \mathcal{B}_1(X)$ and hence $F = \{x;\ f(x) > 0\} \in \mathcal{Z}^\sigma$. Thus we can apply 2.3.)

Corollary 3.7. In every topological space $(X,\mathcal{T})$, the following are equivalent:

(i) $\mathcal{C}_1(X) \subset \mathcal{B}_1(X)$,

(ii) $\mathcal{B}_1^*(X) \subset \mathcal{B}_1(X)$,

(iii) $\mathcal{C}_1(X) \subset \mathcal{A}_1(X) = \mathcal{B}_1(X)$,

(iv) $\mathcal{B}_1^*(X) \subset \mathcal{A}_1(X) = \mathcal{B}_1(X)$,

(v) X is perfect, every locally F_σ-set is F_σ and $\mathcal{Z}^\sigma$ contains the closed sets.

4. Absolute Baire 1 functions.

Let (X,d) be a metric space. The function $f:X \to \mathbb{R}$ is said to be absolute Baire 1 if, for every metric space Y containing X as a subspace, there exists a $g \in \mathcal{B}_1(Y)$ such that $g|_X = f$.

Theorem 4.1. Let (X,d) be a metric space and let $f:X \to \mathbb{R}$ be given. Then the following are equivalent:

(i) f is absolute Baire 1.

(ii) There exists a complete metric space Y containing X as a subspace and $g \in \mathcal{B}_1(Y)$ such that $g|_X = f$.

(iii) $f \in \mathcal{B}_1^*(X)$.

(iv) For every $\epsilon > 0$ and every countable and completely bounded $H \subset X$ there exists an open $G \subset X$ such that $H \cap G \neq \emptyset$ and $\omega(f, H \cap G) < \epsilon$.

Let $\varkappa(X)$ denote the set of functions $f:X \to \mathbb{R}$ such that, for every non-empty, compact $K \subset X$, the restriction $f|_K$ has a continuity point. Then Theorem 4.1 easily implies the following assertion [7].

Corollary 4.2. If (X,d) is a complete metric space then $\mathcal{C}_1(X) = \mathcal{B}_1(X) = \varkappa(X)$.

5. More classes on metric spaces.

Let X be a non-empty set and let $\{f_\alpha\}_{\alpha<\omega_1}$ be a transfinite sequence of real valued functions defined on X. We say that $f:X \to \mathbb{R}$ is the limit of the sequence $\{f_\alpha\}_{\alpha<\omega_1}$ if for every $x \in X$ and $\varepsilon > 0$ there exists $\alpha_0 < \omega_1$ such that $|f_\alpha(x) - f(x)| < \varepsilon$ for every $\alpha_0 < \alpha < \omega_1$. (This is equivalent to the condition that $f_\alpha(x) = f(x)$ for $\alpha > \alpha_0 = \alpha_0(x)$.)

This notion is due to Sierpiński who proved in [18] that the limit of a transfinite sequence of Baire 1 functions is Baire 1. (We remark that the limit of a transfinite sequence of continuous function is continuous; while, supposing the continuum hypothesis, every function is the limit of a transfinite sequence of Baire 2 functions.)

Let $S(X)$ denote the set of limits of transfinite sequences from $\mathcal{B}_1(X)$.

Let $\mathfrak{M}(X)$ denote the set of functions $f:X \to \mathbb{R}$ such that for every countable $H \subset X$ there exists $g \in \mathcal{B}_1(X)$ with $g|_H = f$.

Proposition 5.1. For every metric space (X,d), $S(X) \subset \mathfrak{M}(X)$. If $|X| \leq \aleph_1$ then $S(X) = \mathfrak{M}(X)$.

Let $\mathfrak{B}^s(X)$ denote the set of functions $f:X \to \mathbb{R}$ such that $f|_Y \in \mathfrak{B}(Y)$ for every separable subspace $Y \subset X$.

Theorem 5.2. In every metric space (X,d), we have

$$\mathcal{C}_1(X) \subset \mathcal{B}_1^*(X) \subset \mathcal{A}_1(X) = \mathcal{B}_1(X) \begin{array}{c} \subset S(X) \subset \mathfrak{M}(X) \subset \\ \subset \mathfrak{B}(X) \subset \end{array} \mathfrak{B}^s(X) \subset \varkappa(X).$$

<u>Corollary 5.3</u>. If the metric space (X,d) is complete then the classes above coincide.

In particular, it follows that in complete metric spaces $S(X) = \mathcal{B}_1(X)$. This generalizes the theorem by Sierpiński mentioned above. See also [17].

<u>Corollary 5.4</u>. If (X,d) is a separable metric space then

$$\mathcal{C}_1(X) \subset \mathcal{B}_1^*(X) \subset \mathcal{A}_1(X) = \mathcal{B}_1(X) \subset S(X) \subset \mathcal{M}(X) \subset \mathcal{D}(X) \subset \kappa(X).$$

Let Q denote the set of rational numbers (as a subspace of $\mathbb{R}$). It is easy to see that $\mathcal{C}_1(Q) \neq \mathcal{B}_1^*(Q)$ and $\mathcal{B}_1^*(Q) \neq \mathcal{B}_1(Q)$. Indeed, if $f(\frac{p}{q}) \overset{\text{def}}{=} \frac{1}{q}$ $((p,q) = 1, q > 0)$ and $g(\frac{p}{q}) \overset{\text{def}}{=} q$ $((p,q) = 1, q > 0)$ then $f \in \mathcal{B}_1^*(Q) \setminus \mathcal{C}_1(Q)$ and $g \in \mathcal{B}_1(Q) \setminus \mathcal{B}_1^*(Q)$.

Now let X be a separable metric space such that $|X| = \aleph_1$ and every countable subset of X is G_δ (see [11], p. 517). It is easy to check that in this space $S(X)$ contains <u>every</u> function $f: X \to \mathbb{R}$. If $2^{\aleph_0} = \aleph_1$ then there is a function on X which is not Baire 1 since the number of Baire 1 functions is at most $2^{\aleph_0} = \aleph_1$ while the number of all real valued functions on X is $\aleph_1^{\aleph_1}$. Therefore, supposing the continuum hypothesis, we can find a separable space with $\mathcal{B}_1(X) \neq S(X)$.

We can find other spaces for which $S(X) \neq \mathcal{M}(X)$, $\mathcal{M}(X) \neq \mathcal{D}(X)$ and $\mathcal{D}(X) \neq \varkappa(X)$ holds, respectively. All these examples are based on some "singular" spaces and use the axiom of choice or some hypotheses independent of the axioms of set theory. The following theorem explains, why it is impossible to find some simple, constructive examples.

Let (X,d) be a metric space. The subset $H \subset X$ is called analytic if H is the result of the $(\mathcal{A})$-operation applied to a system of closed subsets of X. The space (X,d) is called absolute analytic if X is analytic in every metric space Y containing X as a subspace. It is well-known that X is absolute analytic if and only if there is a complete metric space Y containing X as an analytic subset (see [9], p. 346).

<u>Theorem 5.5</u>. If (X,d) is absolute analytic then $\mathcal{B}_1(X) = \varkappa(X)$. (See [7], p. 496.)

<u>Corollary 5.6</u>. If (X,d) is absolute analytic then $\mathcal{B}_1(X) = S(X) = \mathcal{M}(X) = \mathcal{D}(X) = \varkappa(X)$.

The space Q (as a Borel subspace of $\mathbb{R}$) is absolute analytic. As we saw, in this space neither $\mathcal{C}_1(X) = \mathcal{B}_1^*(X)$ nor $\mathcal{B}_1^*(X) = \mathcal{B}_1(X)$ is valid.

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