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Herbert Leinfelder*, California Institute of Technology, Department of Mathematics, Pasadena, CA 91125

A Unifying Principle in Real Analysis

This note describes a unifying principle that can be used to deduce several fundamental theorems in real analysis. It involves the concept of a subordering of the real numbers and states that a subordering that is locally valid with respect to each point of an interval I coincides with the usual order on I. A number of applications are given followed by an extended version of the new theorem.

First we define the concept of a subordering and introduce some notation. Let I be an interval in $\mathbb{R}$. A transitive relation $\mathcal{R}$ that preserves the natural order $<$ will be called a *subordering* on I, i.e., $\mathcal{R}$ is a subset of $I \times I$ satisfying:

(A1) If $x \mathcal{R} y$ and $y \mathcal{R} z$ then $x \mathcal{R} z$.

(A2) If $x \mathcal{R} y$ then $x < y$.

Here we use the standard notation $x \mathcal{R} y$ if $(x,y) \in \mathcal{R}$; and we write $x \not\mathcal{R} y$ if $(x,y) \notin \mathcal{R}$. We call a subordering $\mathcal{R}$ *locally valid (with respect to each point* c *of* I*)* if:

$$(A3)\quad \begin{cases} \text{For every } c \text{ in } I \text{ there is a deleted neighborhood} \\ \dot{V}(c) \text{ of } c \text{ such that when } x \text{ belongs to } \dot{V}(c) \\ \text{then } x \mathcal{R} c \text{ or } c \mathcal{R} x . \end{cases}$$

As usual, if $V(c)$ is any neighborhood of c in I (say $V(c) = J \cap I$ for some open interval J containing c) the deleted neighborhood $\dot{V}(c)$ is defined to be $V(c) \setminus \{c\}$.

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* On leave from Mathematisches Institut der Universität Bayreuth, D-8580 Bayreuth, West Germany.

At this point we make two conventions: First, every relation $\mathcal{R}$ appearing in this note is always assumed to preserve the natural ordering, i.e., $\mathcal{R}$ is assumed tacitly to be a subset of $<$ without mentioning this explicitly. Second, all functions in this note are defined on intervals.

Example 1. Let $f\colon I \to \mathbb{R}$ be a differentiable function satisfying $f' > 0$. Take the subordering $\mathcal{R}$ on I to be

$$x \mathcal{R} y \Leftrightarrow f(x) < f(y) \quad \text{if} \quad x < y .$$

Clearly, $\mathcal{R}$ is locally valid. In fact, we have $\frac{f(x)-f(c)}{x-c} > 0$ for all x in some deleted neighborhood $\dot{V}(c)$, because $f'(c) > 0$.

Example 2. Let $(\Omega_\alpha)_{\alpha \in \mathcal{G}}$ be any open covering of the compact interval I and take the subordering $\mathcal{R}$ to be

$$x \mathcal{R} y \Leftrightarrow \begin{cases} \text{the compact interval } [x,y] \text{ has the finite} \\ \text{covering property with respect to } (\Omega_\alpha)_{\alpha \in \mathcal{G}} . \end{cases}$$

Since each point c in I has an (interval) neighborhood $V(c)$, which is contained in some Ω_α, the subordering $\mathcal{R}$ is actually locally valid.

Example 3. Let $f\colon I \to \mathbb{R}$ be continuous and nonzero on I. Then, it is clear that

$$x \mathcal{R} y \Leftrightarrow \operatorname{sgn} f(x) = \operatorname{sgn} f(y) \quad \text{if } x < y$$

generates a locally valid subordering.

From these examples one is tempted to believe that many different, locally valid suborderings exist. But appearances are deceptive.

Theorem 1. *The natural order* $<$ *is the only subordering that is locally valid on the interval* I.

(The proof that follows is based on the nested interval axiom. It is just as easy to base a proof of Theorem 1 on the least upper bound property (see Theorem 2).)

Proof. Let $\mathcal{R}$ be any locally valid subordering on I and suppose there exist points x and y in I satisfying $x < y$ as well as $x \not\mathcal{R} y$. We consider the point $\frac{x+y}{2}$ and observe that not both relations, $x \mathcal{R} \frac{x+y}{2}$ and $\frac{x+y}{2} \mathcal{R} y$, can hold. Hence, we can find points x_1 and y_1 in I obeying $x_1 \not\mathcal{R} y_1$, $[x_1,y_1] \subset [x,y]$ and $y_1 - x_1 = \frac{y-x}{2}$. Continuing this process, we obtain a sequence $([x_n,y_n])_{n \in \mathbb{N}}$ of nested intervals with the property

$$\text{(1)} \qquad x_n \not\mathcal{R} y_n \,, \quad x_n < y_n \,, \quad y_n - x_n = \frac{y-x}{2^n} \,.$$

Let c be the unique point that belongs to all $[x_n,y_n]$. Since $\mathcal{R}$ is locally valid with respect to c,

$$\text{(2)} \qquad x \mathcal{R} c \quad \text{or} \quad c \mathcal{R} x$$

holds true for all x in some deleted neighborhood $\dot{V}(c)$ of c. Moreover, we have

$$\text{(3)} \qquad [x_n,y_n] \subset V(c)$$

for sufficiently large n. If $c = x_n$ or $c = y_n$, we conclude that $x_n \mathcal{R} y_n$ using (2) and (3). Otherwise $x_n < c < y_n$, but then again by (2) and (3) it follows that $x_n \mathcal{R} c$ as well as $c \mathcal{R} y_n$. So in any case we deduce $x_n \mathcal{R} y_n$, contradicting (1). Thus, we have proved Theorem 1. □

Taking Theorem 1 into account, we now review the three examples given above. First of all, we state that we have proved in Example 1 the monotonicity of f, in Example 2 the Heine-Borel theorem, and in Example 3 Bolzano's "Intermediate Value" theorem. Second, we notice that each of these classical results has been derived from a single principle. Moreover, our principle itself is a simple consequence of the nested interval axiom (or equivalently of the least upper bound property).

The following further examples illustrate that proofs based on Theorem 1 are often simpler than proofs based on other principles.

Example 4. Let $f:[a,b] \to \mathcal{R}$ be a regulated function, i.e., a function having one-sided limits $f(c+)$ and $f(c-)$ at each point c in $[a,b]$. Consider the locally valid subordering $\mathcal{R}$ on $[a,b]$ given by

$$x \mathcal{R} y \Leftrightarrow f([x,y]) \text{ is bounded.}$$

Our principle yields $a \mathcal{R} b$. Hence, *every regulated function is bounded on compact intervals.*

Example 5. Let $f: I \to \mathbf{R}$ be a differentiable function satisfying $m < f' < M$ for some constants m,M. Consider the subordering $\mathcal{R}$ on I generated by

$$x \mathcal{R} y \Leftrightarrow m < \frac{f(y)-f(x)}{y-x} < M \quad \text{if} \quad x < y$$

$$\Leftrightarrow m(y-x) < f(y) - f(x) < M(y-x) \text{ if } x < y .$$

Since $m < f'(c) < M$, we conclude that $m < \dfrac{f(x)-f(y)}{x-c} < M$ for all x in some deleted neighborhood $\dot{V}(c)$ of c. Applying Theorem 1 we see that $m < \dfrac{f(y)-f(x)}{y-x} < M$ if $x < y$. Hence, we have proved the *mean value inequality.*

Example 6. Let $I = [a,b]$ be a compact interval and let f be a continuous (or upper-semicontinuous) function on I that does not attain tis maximum. Then, for each c in I a point c' in I exists satisfying $f(c) < f(c')$. By continuity of f there is a neighborhood $V(c)$ of c such that $f(x) < f(c')$ for x in $V(c)$. Thus, we are motivated to introduce a locally valid subordering $\mathcal{R}$ on I given by

$$x \mathcal{R} y \Leftrightarrow \text{there is a } d \in f(I) \text{ satisfying } f < d \text{ on } [x,y] .$$

Theorem 1 yields $a \mathcal{R} b$, which is impossible. It follows that a *continuous* *(or upper semicontinuous) function on a compact interval has a maximum value.*

Example 7. Let f be a continuous function from an interval I into $\mathbb{R}$. For any positive number ε we consider the relation $\mathcal{R}$ defined by

$$x \,\mathcal{R}\, y \Leftrightarrow \begin{cases} \text{there is a } \delta > 0 \text{ such that } |f(x') - f(y')| \leq \varepsilon \\ \text{for all } x', y' \in [x,y] \text{ satisfying } |x'-y'| \leq \delta. \end{cases}$$

Clearly, $\mathcal{R}$ is a locally valid relation. Bearing this in mind, it is easy to see that $\mathcal{R}$ is transitive, i.e., a locally valid subordering. Hence, a *continuous function is uniformly continuous on every compact interval.*

Example 8. Let f be a continuous function on the compact interval I. For any fixed positive integer n we consider the subordering $\mathcal{R}$ on I given by

$$x \,\mathcal{R}\, y \Leftrightarrow \begin{cases} \text{there is a piecewise-linear function} \\ \varphi : [x,y] \to \mathbb{R} \text{ satisfying } \varphi(x) = f(x), \\ \varphi(y) = f(y) \text{ and } |\varphi - f| \leq \frac{1}{n} \text{ on } [x,y]. \end{cases}$$

Since f is continuous at c, we have $|f - f(c)| \leq \frac{1}{2n}$ on some neighborhood $V(c)$ of c. Let $x \in V(c)$ and assume $x < c$ (the case $x > c$ is similar). Consider the linear function $\varphi : [x,c] \to \mathbb{R}$ determined by the conditions $\varphi(x) = f(x)$, $\varphi(c) = f(c)$. Then, $|\varphi - f| \leq \frac{1}{n}$ on $[x,c]$ and thus $x \,\mathcal{R}\, c$. This shows that $\mathcal{R}$ is a locally valid subordering. Applying Theorem 1 we see that a *continuous function on a compact interval can be approximated arbitrarily closely by a piecewise-linear function.*

Example 9. Similarly let f be a regulated function defined on a compact interval I. We will show that f can be approximated uniformly from above by a decreasing sequence of step functions. Following standard arguments and using the boundedness of $f(I)$ as a starting point it suffices to show: Given a positive integer n and a step function $\alpha : I \to \mathbb{R}$ satisfying $f \leq \alpha$ there is a step function $\varphi: I \to \mathbf{R}$ such that $f \leq \varphi \leq \alpha$ and $|f-\varphi| \leq \frac{1}{n}$.

Hence, we introduce the subordering $\mathfrak{R}$ on I

$$x \,\mathfrak{R}\, y \Leftrightarrow \begin{cases} \text{there is a step function } \varphi: [x,y] \to \mathbf{R} \text{ which} \\ \text{satisfies } f \leq \varphi \leq \alpha \text{ and } |f-\varphi| \leq \frac{1}{n} \text{ on } [x,y] . \end{cases}$$

Using the fact that f has one-sided limits it is not hard to show that $\mathfrak{R}$ is locally valid. Thus, a *regulated function can be approximated uniformly from above by a decreasing sequence of step functions.*

Example 10. Finally, let f be a continuous real-valued function. Denote by $J^{(\pm)}(f;x,y)$ the upper (lower) integral of f over $[x,y]$ divided by $y - x$. For any positive ϵ

$$x \,\mathfrak{R}\, y \Leftrightarrow -\epsilon \leq J^{+}(f;x,y) - J^{-}(f;x,y) \leq \epsilon$$

defines a locally valid subordering. Hence, a *continuous function is integrable over every compact interval.*

In our opinion the suborderings described above are the most interesting ones to which Theorem 1 applies. Of course, there are other more or less natural examples. For instance, consider a closed differential form $\omega = Pdx + Qdy$ with continuous coefficients P and Q in an open set $\Omega \subset \mathbf{R}^2$ and a continuous curve γ supported in Ω (compare [1;pp.56]). Then

$$x \,\mathfrak{R}\, y \Leftrightarrow \omega \text{ has a primitive along } \gamma\,|\,[x,y]$$

defines a locally valid subordering on the domain of γ.

As a matter of fact, Theorem 1 shows that a "property $\mathfrak{R}$" concerning subintervals of a given interval is valid for *all* compact subintervals if

i) "property $\mathfrak{R}$" is *hereditary*, i.e., if both $[x,y]$ and $[y,z]$ have "property $\mathfrak{R}$", then $[x,z]$ has.

ii) "property $\mathfrak{R}$" is *locally valid.*

Looking at all those examples presented above, we believe that it is worthwhile to figure out what is the essence of Theorem 1. To do so, let us consider a *chain* $(X,<)$, i.e., a set X linearly ordered by an antisymmetric relation $<$ [2, p.15]. We endow X with the *order-topology* which has a subbase consisting of all sets of the form $\{x \mid x < a\}$ or $\{x \mid a < x\}$ for some a in X. As in the case $X = \mathbf{R}$ we call a relation $\mathfrak{R}$ on X a locally valid subordering if $\mathfrak{R}$ satisfies $(A1)$ - $(A3)$.

Theorem 2. *A chain* $(X,<)$ *endowed with the order-topology is connected iff* $<$ *is the only subordering that is locally valid on* X.

Proof. Let $x,y \in X$, $x < y$ and assume that X is connected, i.e., X is order-complete and has no gaps [2,pp. 57-58]. Define the non-empty set $A = \{z \mid x \,\mathfrak{R}\, z \text{ and } x \leq z \leq y\}$ and consider $\sup A$. In view of $(A1)$ and $(A3)$ we have $\sup A = y$. Using $(A3)$ again, we finally get $x \,\mathfrak{R}\, y$. On the other hand, if X is not connected, then X can be written as a disjoint union of two non-empty sets A_1, A_2. Define the locally valid sub-ordering $\mathfrak{R}$ by setting $\mathfrak{R} = \{(x,y) \mid x < y;\ x \text{ and } y \text{ belong to the same } A_i\}$. Then $\mathfrak{R}$ is strictly contained in $<$. □

Since most theorems in real analysis rederived in this note actually do not involve orderings, it might be interesting to have a substitute for Theorem 1 and 2 where only topological properties enter. This can be done very easily.

Let X be a topological space. We call an equivalence relation $\sim$ *locally valid* if $(A3)$ is satisfied. Notice that $\sim$ is locally valid iff $\sim$ is *continuous* in the sense that $x' \sim y'$, whenever $x \sim y$ and $x' \in V(x)$, $y' \in V(y)$ for some neighborhoods $V(x)$ and $V(y)$. An equivalence relation is called *trivial* if every two elements are equivalent.

Theorem 3. *A topological space* X *is connected iff every continuous equivalence relation is trivial.*

Proof. If $\sim$ is any continuous equivalence relation on X, then X is the disjoint union of its open equivalence classes $\{y \mid x \sim y\}$. Hence, X is connected iff $\sim$ is trivial. □

The broad applicability of our principle presented above leads us to suppose that it is known in the mathematical literature. But we were unable to find a suitable reference. Fortunately, during a stay at the California Institute of Technology, Professor T. M. Apostol drew my attention to the article [3] . Actually, the article [3] and our note are closely related, but there is also a major difference . The principle presented in [3] is based on an extended Bolzano-Weierstrass theorem, whereas the principle described in this note depends critically on the topological concept of connectedness.

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