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Jack Ceder, Department of Mathematics, University of California at Santa Barbara, Santa Barbara, CA 93106

Some Problems on Borel 1 Selections

A set-valued function ϕ from R^n into the non-void subsets of R^m (written $\phi : R^n \to 2^{R^m}$) has a Borel 1 selection f if f is a Borel 1 function such that $f(x) \in \phi(x)$ for all $x \in R^n$.

Below we summarize some of the important results and open problems in the search for Borel 1 selections in the context of Euclidean spaces (for simplicity's sake). [See [2] and [3] for a more detailed treatment as well as for reference documentation.]

By graph ϕ we mean the set $\{(x,y) : y \in \phi(x)\}$ and by $\phi^{-1}(V)$ we mean the set $\{x : \phi(x) \cap V \neq \Phi\}$. We say that ϕ is l.s.c. (1) if $\phi^{-1}(V)$ is an F_σ set whenever V is open.

<u>Theorem 1</u>. <u>Let</u> $\phi : R^n \to 2^{R^m}$. <u>Then</u> ϕ <u>has a Borel 1 selection if any one of the following conditions is satisfied</u>.

(1) (Debs) ϕ is l.s.c. (1) <u>and graph</u> ϕ <u>is a</u> G_δ <u>set</u>.

(2) (Kuratowski, Ryll-Nardzewski) ϕ <u>is</u> l.s.c. (1) <u>and each</u> $\phi(x)$ <u>is closed</u>.

(3) (Ceder, Levi) ϕ <u>is</u> l.s.c. (1) <u>and each</u> $\phi(x)$ <u>is convex</u>.

(4) (Ceder, Levi) graph ϕ <u>is an</u> F_σ <u>set</u>.

(5) (Čoban, Engelking) $\phi^{-1}(F)$ <u>is closed whenever</u> F <u>is closed and each</u> $\phi(x)$ <u>is closed</u>.

Among the open problems for $\phi : R^n \to 2^{R^m}$ are the following:

(a) does ϕ <u>have a Borel 1 selection when</u> ϕ <u>is l.s.c. (1) and graph</u> ϕ <u>is an</u> $F_{\sigma\delta}$ <u>set?</u>

(b) does ϕ have a Borel 1 selection when ϕ is l.s.c. (1) and each $\phi(x)$ is an arc?

(c) does ϕ have a Borel 1 selection when $\phi = f^{-1}$ where f is an open Borel 2 function from R^m into R^n?

There are examples to show that (1) f^{-1} need not have a Borel 1 selection when f is open and (2) ϕ need not have a Borel 1 selection when ϕ is l.s.c. (1) and each $\phi(x)$ is open.

Theorem 2 (Ceder [1]). Let $\phi : R \to 2^R$ and each $\phi(x)$ be closed and connected. Then, ϕ has a Borel 1 selection if and only if for each nowhere dense perfect set P, $\phi|P$ (the restriction of ϕ to P) has a Borel 1 selection.

Examples show that the above result does not hold for open intervals nor does it have any direct analogue for Borel 2 selections. Some interesting problems suggested by this result are the following:

(α) can Theorem 2 be extended to apply to $\phi : R^n \to 2^{R^m}$.

(β) can the connectedness of the values $\phi(x)$ be dropped.

(γ) does Theorem 2 work for perfect sets of measure 0.

Bibliography

(1) J. Ceder, On Baire 1 Selections, Ricerche Mat. 30 (1981), 305-315.

(2) J. Ceder, S. Levi, On the Search for Borel 1 Selections (to appear).

(3) J. Ceder, On Some Questions on Borel 1 Selections (in preparation).

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