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Some Counterexamples in Multifunction Theory

If F is a function, called a multifunction, from a given topological space T into the space $P(Y)$ of all nonempty subsets of a given topological space Y, then a selection for F is any function $f : T \to Y$ such that $f(t) \in F(t)$ for all $t \in T$. The problem of finding nice selections for nice multifunctions is an old one and has been extensively studied. However, most of the work has been devoted to finding selections which are measurable with respect to some measure. (See [12] for a list of references.) If $U \subset Y$, then $F^{-}(U)$ is defined as usual for relations, so that

$$F^{-}(U) = \{t \in T : (y,t) \in F^{-1} \text{ for some } y \in U\} = \{t \in T : F(t) \cap U \neq \emptyset\}.$$

The conditions under which a multifunction F admits a Borel 1 selection in terms of the topological nature of the values $F(t)$, of the graph $\mathrm{Gr}\, F = \{(t,y) : y \in F(t)\}$ and of the inverse image $F^{-}(U)$ of open sets U under F were recently collated and investigated in [1]. The aim of this note is to answer some problems posed in [1]. The results given below are inspired by corresponding results obtained in [3], [7], and [8].

By R we mean the set of all real numbers. By $\mathcal{F}(Y)$ (resp. $\mathcal{K}(Y)$) we mean the classes of all closed (resp. convex) members of $P(Y)$. If $F : T \to P(Y)$, we say that F is lower semicontinuous (briefly lsc), if $F^{-}(U)$ is open in T, whenever U is open in Y. If we impose the condition that the set $F^{-}(U)$ be of Borel additive class α, then we say that F is lower semicontinuous of class α or lsc α. Notice that when $F(t) = \{f(t)\}$ for a function $f: T \to Y$, the definitions of lsc α and lsc reduce to the definitions of a Borel α and continuous functions respectively. Rather stringent conditions must be imposed upon a multifunction to admit a continuous selection. The most sifnificant result in this regard is the following.

<u>Theorem 1</u> ([11]). Let $F: T \to \mathcal{K}(R^n)$. If T is perfectly normal and F is lsc, then F has a continuous selection.

In [10] the following general problem is formulated:

<u>Problem</u>: Under what assumptions can theorems known for $\alpha = 0$ (i.e. theorems on lsc multifunctions) be extended to arbitrary $\alpha < \Omega$?

Paper [1] contains the following Borel α analogue of Michael's Theorem 1.

<u>Theorem 2</u> ([1]). Let $F: X \to \mathcal{K}(R^n)$ where X is a metric space. If F is lsc α, then F has a Borel α selection.

We may ask whether the range space in Theorem 2 can be generalized. Specifically, [1] asks the following questions.

QUESTION 6 (original numeration). Let $F : T \to \mathcal{K}(Y)$ where T is a metric and Y is an infinite dimensional normed linear space. Does F have a Borel 1 selection when F is lsc 1?

QUESTION 7. Let $F : T \to P(R^n)$ where T is a metric space. Does F have a Borel 1 selection when F is lscl and each $F(t)$ is an arc?

The following four counterexamples show that the answer to both questions is negative, even if F is lsc.

EXAMPLE 1. Let $T := [0;1]$ and let Y be the separable Banach space of continuous real-valued functions on the interval T with the sup norm. There exists a lsc multifunction $F : T \to \mathcal{K}(Y)$, which admits no (Borel) measurable selection.

Proof: Decompose T into 2 disjoint nonmeasurable subsets A and B. The set H of polynomials with real coefficients is a dense, F_σ, non-complete, linear subspace of Y. Define a multifunction $F : T \to \mathcal{K}(Y)$ by setting

$$F(t) = \begin{cases} H & \text{if } t \in A \\ H + \exp(\cdot) & \text{if } t \in B. \end{cases}$$

The multifunction F is lsc since $F^-(U)$ is T or empty for every open subset U of Y. Clearly F is convex-valued. However, if f is any selection for F, then $f^-(H)$ is non-measurable while H is a F_σ-set. That completes the proof.

In the following example values of F are 1-dimensional convex sets.

EXAMPLE 2. There exists a nonseparable prehiblert space Y and a lsc 1 multifunction $F : R \to \mathcal{K}(Y)$ which has no measurable selection.

Proof: Define $Y := \{h : R \to R : \text{supp } h := \{x : h(x) \neq 0\} \text{ is finite}\}$. Then

$$\langle g | h \rangle := \sum_{x \in R} g(x) \cdot h(x)$$

is a scalar product in Y. Decompose R into 2 disjoint non-measurable subsets: $R = A \cup B$ and put

$$F(t) = \begin{cases} \{g \in Y : g(t) > 0 \text{ and } \text{supp } g = \{t\}\} & \text{if } t \in A \\ \{g \in Y : g(t) < 0 \text{ and } \text{supp } g = \{t\}\} & \text{if } t \in B. \end{cases}$$

Notice that F is lsc 1. Next note that $Z := \{g \in Y : g(x) \geq 0$ for all $x \in R\}$ is closed in Y. If $f(t) \in F(t)$ for all $t \in R$, then $f^{-}(Z) = A$. Hence f is nonmeasurable.

EXAMPLE 3. There exists a lsc multifunction $F : [0, 2\pi) \to P(R^n)$, $n \geq 2$ with open arcs as values but having no Borel 1 selection.

Proof: Let $S := \{x_1, x_2, x_3, \ldots, x_n) : x_1^2 + x_2^2 = 1, x_3 = \cdots = x_n = 0\}$ denote the unit circle in R^n. Since the class of all Borel functions from $T := [0; 2\pi)$ to S has the same cardinality as T, it is clear that we can choose a function $h : T \to S$ such that the graph of h intersects the graph of each Borel function from T to S. For each $t \in T$ put

$$F(t) := S - \{h(t)\}.$$

Obviously, each $F(t)$ is an arc. It is easy to show that F is lsc, since $F^{-}(U)$ is empty or T for open $U \subset R^n$. However, if $f : T \to S$ is any Borel function, then there exists a $t_0 \in T$ such that $f(t_0) = h(t_0)$. Hence F admits no Borel 1 selection.

Remark: Example 3 also shows that a lsc multifunction with open, connected values may fail to have a Borel graph.

EXAMPLE 4. There exists a lsc multifunction $F : [0, 2\pi) \to R^n$, $n \geq 2$ with a $G_{\delta\sigma}$ graph and arcwise connected values but, with no Borel 1 selection.

Proof: Define T and S as in the proof of example 3.Z.Grande recently exhibited a Borel 2 function $g : T \to T$ which intersects each Borel 1 function from T into T (see [5]). Obviously, by the lifting theorem, for every map $f : T \to S$ there exists a map $h : T \to T$ such that

$$f(t) = (\cos h(t), \ \sin h(t), \ 0, \ldots, 0) \in S.$$

Put

$$F(t) = S - \{(\cos g(t), \sin g(t), 0, \ldots, 0)\}.$$

It is easily checked that F has a $G_{\delta\sigma}$ graph but no Borel 1 selection.

In [8] a general theorem on selectors was proved (cf. also [4]) which implies that for a metric space T, a complete separable metric space Y and every lsc $F: T \to \mathcal{F}(Y)$ there exists a Borel 1 selection. A further interesting question (see also [2], [4], [6], [8]) is whether or not the Polish-ness of Y in this theorem is essential. Specifically:

<u>QUESTION 4</u>. Do there exist metric spaces T and Y and a lsc 1 multifunction $F: T \to \mathcal{F}(Y)$ which has no Borel 1 selection?

<u>EXAMPLE 5</u>. Assuming the continuum hypothesis, there exist metric spaces T and Y and a lsc 1 multifunction $F: T \to \mathcal{F}(Y)$ which has no Borel selection.

<u>Proof</u>: (cf [8]): Let T be the unit interval with the usual metric and let $Y = \{\alpha : \alpha < \Omega\}$ be the set of all countable ordinals with the discrete metric. Arrange all points of T in a transfinite sequence of type Ω and define

$$T \ni t_\alpha \to F(t_\alpha) = \{\beta : \beta \geq \alpha\} \quad \text{for} \quad \alpha < \Omega.$$

Evidently, F is lsc 1 closed-valued multifunction. Let $f: T \to Y$ be any selection for F. Since $f(t_\alpha) \geq \alpha$ for every $\alpha < \Omega$, each fiber $f^{-}(\{\delta\})$, $\delta < \Omega$, is countable. Since the fibers are pairwise disjoint and since the family of all subsets of all fibers has cardinality greater than the set of all Borel sets, it follows that there exists a $Z \subset Y$ so that $f^{-}(Z) = \bigcup_{\delta \in Z} f^{-}(\{\delta\})$ is nonmeasurable. Since Z is closed, this completes the argument.

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