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MULTIPLIERS OF SUMMABLE DERIVATIVES

Theorem 8 of this note characterizes the system of all functions g such that the product fg is a derivative for each summable derivative f. If we require the product fg to be a summable derivative, we get the same system.

In this way we obtain a solution of Problem 4.1 posed in [1] by R.J. Fleissner.

The word function means throughout this note a (finite) real function defined on a subset of $R = (-\infty,\infty)$. For each interval J let $D(J)$ be the system of all finite derivatives on J.

Let $a,b \in R$, $a < b$. Let g be a function defined on a set containing the interval $J = [a,b]$ and let m be a natural number. By $v(m,J,g)$ or $v(m,a,b,g)$ we shall denote the least upper bound of the set of all sums $\sum_{k=1}^{m} |g(y_k) - g(x_k)|$, where $a \leqq x_1 < y_1 \leqq x_2 < y_2 \leqq \cdots \leqq x_m < y_m \leqq b$. Note that $v(1,J,g)$ is the oscillation of g on J, $v(m,J,g) \leqq v(m+1,J,g)$ for each m and that $\lim_{m\to\infty} v(m,J,g)$ is the variation of g on J.

We shall keep the meaning of the symbols a, b, J, m throughout sections 1-3. The integrals are Perron integrals.

1. Let $g \in D(J)$, $T \in (-\infty, |g(b) - g(a)|)$. Then there is a function f piecewise linear on J such that $f(a) = f(b) = \int_J f = 0$, $\int_J |f| = 2$ and $\int_J fg > T$.

Proof: Let, e.g., $g(a) \geqq g(b)$. Choose an $\varepsilon \in (0,\infty)$ such that $g(a) - g(b) - 4\varepsilon > T$. Set $s = (a+b)/2$. There is a $c \in (a,s)$ such that $\int_a^c g > (c-a)(g(a) - \varepsilon)$. There is a $\delta \in (0,\infty)$ such that $a + \delta < c$, $c + \delta < s$ and that $|\int_a^x g| + |\int_c^y g| < \varepsilon(c-a)$, whenever $x \in [a, a+\delta]$ and $y \in [c, c+\delta]$. Set $Q = 1/(c-a)$. Let p be a function on J with the following properties: $p = 0$ on $\{a\} \cup [c+\delta, b]$, $p = Q$ on $[a+\delta, c]$, p is linear on $[a, a+\delta]$ and on $[c, c+\delta]$. Obviously $\int_J p = 1$. Set $A = \int_a^{a+\delta} (p-Q)g$, $C = \int_c^{c+\delta} pg$. Then $\int_J pg = Q \int_a^c g + A + C$. It follows from the second mean value theorem that there is an $x \in [a, a+\delta]$ and a $y \in [c, c+\delta]$ such that $A = -Q \int_a^x g$, $C = Q \int_c^y g$. Hence $\int_J pg > g(a) - 2\varepsilon$. In a similar way we construct a nonnegative piecewise linear function q on J such that $q = 0$ on $[a,s] \cup \{b\}$, $\int_J q = 1$ and that $\int_J qg < g(b) + 2\varepsilon$. Now we set $f = p - q$.

2. Let $g \in D(J)$, $T \in (-\infty, v(m,J,g))$. Then there is a piecewise linear function f on J such that

$\int_J |f| = 2m$, $|\int_a^x f| \leq 1$ for each $x \in J$, $\int_J f = 0$ and $\int_J fg > T$.

(This follows easily from 1.)

3. Let f and g be measurable functions on J. Let $\int_J |f| < \infty$ and let g be bounded. Set $A = \max\{|\int_a^x f|; x \in J\}$, $B = v(m,J,g)$. Then

$$|\int_J fg| \leq \frac{B}{m} \int_J |f| + A(B + |g(b)|).$$

Proof: Set $C = \int_J |f|$. There are $y_k \in J$ such that $a = y_0 < y_1 < \cdots < y_m = b$ and that $\int_{y_{k-1}}^{y_k} |f| = C/m$. Set

$s_k = \sup\{|g(y_k) - g(x)|; y_{k-1} < x < y_k\}$ $(k = 1,\ldots,m)$,

$P = \sum_{k=1}^m \int_{y_{k-1}}^{y_k} f\cdot(g - g(y_k))$, $Q = \sum_{k=1}^m g(y_k) \int_{y_{k-1}}^{y_k} f$.

Obviously $|P| \leq \sum_{k=1}^m s_k \int_{y_{k-1}}^{y_k} |f| = \frac{C}{m} \sum_{k=1}^m s_k$. Let $\varepsilon \in (0,\infty)$. There are $x_k \in (y_{k-1}, y_k)$ such that $|g(y_k) - g(x_k)| > s_k - \varepsilon$. Since $\sum_{k=1}^m |g(y_k) - g(x_k)| \leq B$, we have $\sum_{k=1}^m s_k \leq B + m\varepsilon$ so that $|P| \leq C(\frac{B}{m} + \varepsilon)$, $|P| \leq CB/m$. Since $Q = \sum_{k=1}^{m-1}(g(y_k) - g(y_{k+1})) \int_a^{y_k} f + g(y_m) \int_a^{y_m} f$, we have $|Q| \leq A(B + |g(b)|)$. Now we note that $\int_J fg = P + Q$.

4. Let f and g be measurable functions on the interval $[0,1]$. Let $\int_0^1 |f| < \infty$, $\frac{1}{x} \int_0^x f \to 0$ $(x \to 0+)$ and let g be bounded. For each natural number n set

$V_n = v(2^n, 2^{-n}, 2^{-n+1}, g)$. Suppose that $\sup_n V_n < \infty$. Then $\frac{1}{x}\int_0^x fg \to 0 \ (x \to 0+)$.

Proof: Set $x_k = 2^{-k}$ $(k = 0,1,\ldots)$, $S = \sup\{|g(x)|;\ x \in [0,1]\}$, $V = \sup_n V_n$. Let $\varepsilon \in (0,\infty)$. Set $\delta = \varepsilon/(2V + S + 1)$. There is a natural number r such that $\int_0^{x_r} |f| < \delta$ and that $3|\int_0^x f| \leqq \delta x$ for each $x \in (0, x_r]$. If $k > r$ and if $x_k < x \leqq 2x_k$, then $|\int_{x_k}^x f| \leqq \frac{\delta}{3}(x + x_k) \leqq \delta x_k$ so that, by 3 with $m = 2^k$, $|\int_{x_k}^x fg| \leqq x_k V_k \delta + \delta x_k (V_k + S) \leqq x_k \delta (2V + S) \leqq x_k \varepsilon$. Now let $x \in (0, x_r]$. There is an $n > r$ such that $x_n < x \leqq 2x_n$ and, by what has just been proved, $|\int_0^x fg| \leqq \sum_{k=n+1}^{\infty} |\int_{x_k}^{2x_k} fg| + |\int_{x_n}^x fg| \leqq \sum_{k=n}^{\infty} \varepsilon x_k = 2\varepsilon x_n < 2\varepsilon x$. This completes the proof.

5. Let $g \in D([0,1])$. Then

(1) $\limsup_{x \to 0+} g(x) \leqq$

$$\leqq g(0) + \limsup_{n\to\infty} v(1, 2^{-n}, 2^{-n+1}, g)\ .$$

Proof: Let $G' = g$. For $n = 1,2,\ldots$ set $x_n = 2^{-n}$, $J_n = [x_n, 2x_n]$, $s_n = \sup g(J_n)$, $\gamma_n = (G(2x_n) - G(x_n))/x_n$. For each n we have $\gamma_n \geqq \inf g(J_n)$, hence $s_n \leqq \gamma_n + v(1, J_n, g)$. Obviously $\limsup_{n\to\infty} s_n = \limsup_{x\to 0+} g(x)$, $\gamma_n \to g(0)$. This easily implies (1).

6. Notation. Let $J = [0,1]$, $D = D(J)$. By SD we denote the system of all functions $f \in D$ for which $\int_J |f| < \infty$. For each system Q of functions on J let $M(Q)$ be the system of all functions g on J such that $fg \in Q$ for each $f \in Q$. Let Z be the system of all functions g on J such that $fg \in D$ for each $f \in SD$. Let W be the class of all functions g on J such that

$$(2) \qquad \limsup_{n\to\infty} v(2^n, x+2^{-n}, x+2^{-n+1}, g) < \infty \quad \text{for each } x \in [0,1)$$

and

$$(3) \qquad \limsup_{n\to\infty} v(2^n, x-2^{-n+1}, x-2^{n}, g) < \infty \quad \text{for each } x \in (0,1] .$$

Remark. The inequality in (2) is fulfilled, if

$$\limsup_{y\to x+} |(g(y)-g(x))/(y-x)| < \infty .$$

7. Let $g \in D \cap W$. Then g is bounded.

(This follows easily from 5.)

8. We have $Z = D \cap W = M(SD)$.

Proof: I. Let $g \in Z$. It is obvious that $g \in D$. Suppose that, e.g., (2) fails for $x = 0$. Set $V_n = v(2^n, 2^{-n}, 2^{-n+1}, g)$. There are integers r_k such that

$1 < r_1 < r_2 < \dots$ and that $V_{r_k} > k^2$ for each k. Choose a k and set $m = 2^{r_k}$, $a = 1/m$.

Since $v(m,a,2a,g) = V_{r_k}$, there is, by 2, a function h piecewise linear on J such that $h = 0$ on $[0,a] \cup [2a,1]$, $\int_J |h| = 2m$, $\int_J h = 0$, $\int_J hg > k^2$ and that $|\int_0^x h| \leqq 1$ $(x \in J)$. It is easy to see that $|\int_0^x h| \leqq mx$ $(x \in J)$. For each k construct such a and h and set $f_k = ah/k^2$. Further define $f = \sum_{k=1}^{\infty} f_k$. Obviously $\int_J |f| = \sum_{k=1}^{\infty} 2/k^2 < \infty$. If k, a and h are as above and if $x \in [a,2a]$, then $|\int_0^x f| = |\int_0^x f_k| \leqq x/k^2$ and $\int_a^{2a} fg = \int_a^{2a} f_k g > a$. We see that $f \in SD$ and that $fg \notin D$. This contradiction shows that $g \in W$. Hence $Z \subset D \cap W$.

II. Let $g \in D \cap W$ and let $f \in SD$. By 7, g is bounded. Set $f_1 = f - f(0)$. It follows from 4 that $\frac{1}{x}\int_0^x f_1 g \to 0$. Hence $\frac{1}{x}\int_0^x fg \to f(0)g(0)$ $(x \to 0+)$. This shows that $fg \in D$. Obviously $\int_J |fg| < \infty$ whence $fg \in SD$, $g \in M(SD)$, $D \cap W \subset M(SD)$.

III. It is easy to see that $M(SD) \subset Z$. This completes the proof.

9. Let $g \in M(SD)$. Then g is bounded and approximately continuous.

Proof: The boundedness of g follows from 8 and 7. We see, in particular, that $g \in SD$. Therefore $g^2 \in D$. According to a well-known theorem (see, e.g., [1], Theorem 3.3) g is approximately continuous.

Remark. R.J. Fleissner described in [2] the system $M(D)$. His characterization involves the notion of an improper Lebesgue-Stieltjes integral. It is, however, possible to characterize $M(D)$ in the following way which is analogous to our description of $M(SD)$: A function $g \in D$ belongs to $M(D)$ if and only if

$$\limsup_{n\to\infty} \operatorname{var}(x + 2^{-n}, x + 2^{-n+1}, g) < \infty$$

$$\text{for each } x \in [0,1)$$

and

$$\limsup_{n\to\infty} \operatorname{var}(x - 2^{-n+1}, x - 2^{-n}, g) < \infty$$

$$\text{for each } x \in (0,1]$$

(where var ... has the usual meaning). This assertion will be proved elsewhere.

REFERENCES

[1] R.J. Fleissner, Multiplication and the fundamental theorem of calculus: A survey, Real Analysis Exchange, Vol. 2, No. 1 - 1976, 7-34.

[2] ______________, Distant bounded variation and products of derivatives, Fund. Math. XCIV (1977), 1-11.

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