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Associated Sets of Baire* 1 Functions

For a real valued function f defined on a connected subset I of the real line R, the associated sets of f are the sets $E^{\alpha}(f) = \{x : f(x) < \alpha\}$ and $E_{\alpha}(f) = \{x : f(x) > \alpha\}$ for real α. In [3], f is Baire* 1 or $f \in \mathcal{B}_1^*$ if, for every perfect set $P \subset I$, there is a portion of P on which the restriction of f is continuous. The purpose of this paper is to investigate associated sets of these functions. It is shown that neither the class $\mathcal{B}_1^*$ nor the class of Darboux functions in $\mathcal{B}_1^*$ can be characterized in terms of associated sets. However, the family of these sets for each of these two classes can be characterized.

Let $\mathcal{C}$, $\mathcal{B}_1$ and $\mathcal{B}_1^*$ denote the classes of continuous, Baire 1 and Baire* 1 functions on I respectively. Csaszár and Laczkovich proved that $\mathcal{B}_1^*$ is the class of discrete limits of sequences in $\mathcal{C}$ ([1], Corollary 14 and Theorem 15). For a class of functions $\mathcal{F}$, we use $\mathcal{E}(\mathcal{F})$ to denote the family of associated sets of functions in $\mathcal{F}$.

Theorem 1. $E \in \mathcal{E}(\mathcal{B}_1^*)$ if and only if $E \in F_{\sigma} \cap G_{\delta}$.

Proof. Let E be an arbitrary set in $F_\sigma \cap G_\delta$. By Lemma 6 in [1], the characteristic function of E is in $\mathcal{B}_1^*$ and hence $E \in \mathcal{E}(\mathcal{B}_1^*)$. On the other hand, let $f \in \mathcal{B}_1^*$ and $\alpha \in R$ be given. From the proof of Lemma 4.1 in [2], we see that there exist closed sets A_i and $g_i \in \mathcal{C}$ $(i = 1, 2, \cdots)$ such that $I = \bigcup \{ A_i : i = 1, 2, \cdots \}$ and $f|A_i = g_i|A_i$ for each i. It follows that $\{x : f(x) \geq \alpha\}$ and $\{x : f(x) \leq \alpha\}$ are in F_σ. Clearly $E^\alpha(f)$ and $E_\alpha(f)$ are in G_δ. Also, $E^\alpha(f)$ and $E_\alpha(f)$ are in F_σ since $f \in \mathcal{B}_1^* \subset \mathcal{B}_1$. The theorem is proved.

If a class of functions $\mathcal{F}$ can be characterized in terms of associated sets, that is, if there is a family of sets $\mathcal{S}$ such that "$f \in \mathcal{F}$ if and only if $E^\alpha(f)$ and $E_\alpha(f)$ are in $\mathcal{S}$ for $\alpha \in R$" holds, then we must have $\mathcal{E}(\mathcal{F}) \subset \mathcal{S}$.

It is well known that there exists a monotone increasing function g which is not continuous at every rational point. Clearly, for every $\alpha \in R$, $E^\alpha(g)$ and $E_\alpha(g)$ are in $F_\sigma \cap G_\delta$ or $\mathcal{E}(\mathcal{B}_1^*)$ but $g \notin \mathcal{B}_1^*$. Thus $\mathcal{B}_1^*$ cannot be characterized in terms of associated sets.

Now let $\mathcal{D}$ denote the class of Darboux functions on I. $\mathcal{D}\mathcal{B}_1$ and $\mathcal{D}\mathcal{B}_1^*$ are short notations for $\mathcal{D} \cap \mathcal{B}_1$ and $\mathcal{D} \cap \mathcal{B}_1^*$ respectively.

Lemma. Let $f \in \mathcal{D}\mathcal{B}_1^*$, $\alpha \in R$ and E be either $E^\alpha(f)$ or $E_\alpha(f)$. Then $E \in F_\sigma \cap G_\delta$ and satisfies

(*) For $x \in E$ and $\delta > 0$ with $x - \delta \in I$ $(x + \delta \in I)$, the set $[x - \delta, x] \cap E$ $([x, x + \delta] \cap E)$ contains an interval.

Proof. Let f and E be as stated in the hypothesis. We fix $x_0 \in E$ and $\delta > 0$ such that $x_0 - \delta \in I$ $(x_0 + \delta \in I)$. Let J be the interval $[x_0 - \delta, x_0]$ $([x_0, x_0 + \delta])$. Then $f|J$ is a Baire* 1 function and, by Theorem 2 in [3], there exists $x \in J \cap E$ such that f is continuous at x. Clearly $J \cap E$ contains an interval. By Theorem 1, the proof is completed.

Definition. A set of real numbers $E \in SM_2$ if $E \in F_\sigma \cap G_\delta$ and satisfies the condition (*) stated above.

Theorem 2. $E \in \mathcal{E}(\mathcal{X}\mathcal{B}_1^*)$ if and only if $E \in SM_2$.

Proof. The sufficiency follows from the Lemma. To prove the necessity, let $E \in SM_2$ be given. Case $E = \emptyset$ is trivial. We assume that $E \neq \emptyset$ and let E°, $\overline{E}$ and βE denote the interior, the closure and the boundary of E respectively. The condition (*) implies that $E^\circ \neq \emptyset$ and $E \subset \overline{E^\circ}$, the closure of E°. E° is the union of at most countably many disjoint open intervals, say $E^\circ = \bigcup (a_n, b_n)$. Let $c_n = \frac{1}{2}(a_n + b_n)$, $y = L_n(x)$ be the line joining the points $(a_n, 0)$ and $(c_n, 1)$, and $\{x_{nk} : k = 1, 2, \cdots\}$ be a strictly decreasing sequence in the interval (a_n, c_n) converging to a_n. We define f as follows:

$$f(x) = 1 \quad \text{if } x = c_n \text{ or } x_{nk} \text{ for even } k,$$
$$= L_n(x) \quad \text{if } x = x_{nk} \text{ for odd } k,$$
linear on $[x_{n1}, c_n]$ and on $[x_{n\,k+1}, x_{nk}]$ for each k.

By reflecting the graph of f on $(a_n, c_n]$ about the line $x = c_n$, we have f defined on (a_n, b_n). As this is done for each n, f is defined and continuous on E^o. For $x \notin E^o$, let

$$f(x) = 0 \quad \text{if } x \notin E,$$
$$= 1 \quad \text{if } x \in E - E^o.$$

Thus f is defined on R.

Let P be a given perfect set. If $P \cap E^o \neq \emptyset$ or $P \cap (R - \bar{E}) \neq \emptyset$, then there is certainly a portion of P on which the restriction of f is continuous. If $P \cap E^o$ and $P \cap (R - \bar{E})$ are both $\emptyset$, then $P \subset \beta E$. Now $P \cap E$ and $P - E$ are both in G_δ and hence can not be both dense in P. In case $P \cap E$ is not dense in P, there is an interval J_1 with $P \cap J_1 \neq \emptyset$ and $P \cap E \cap J_1 = \emptyset$. It follows that $f|P \cap J_1$ is constantly 0. In case $P - E$ is not dense in P, there is an interval J_2 with $P \cap J_2 \neq \emptyset$ and $(P - E) \cap J_2 = \emptyset$. Then $P \cap J_2 \subset P \cap E \subset \beta E \cap E \subset E - E^o$, and $f|P \cap J_2$ is constantly 1. Therefore $f \in \mathcal{B}_1^*$. From a theorem of Young [4], we have $f \in \mathcal{D}$. Since $E = \{x : f(x) > 0\}$, the theorem is proved.

We will present a function f on R such that, for every $\alpha \in R$, $E^{\alpha}(f)$ and $E_{\alpha}(f)$ are in SM_2 but $f \notin \mathcal{DB}_1^*$. By the remark following Theorem 1, we conclude that $\mathcal{DB}_1^*$ cannot be characterized in terms of associated sets.

Let $\{(a_n, b_n) : n = 1, 2, \cdots\}$ be the contiguous intervals of the Cantor set K in $[0, 1]$. We define f on each (a_n, b_n) in the same manner as in the proof of Theorem 2, and

$$f(x) = 0 \quad \text{if} \quad x \leqq 0 \quad \text{or} \quad x \geqq 1,$$
$$= \sup \{b_n : b_n < x\} \quad \text{if} \quad x \in K \cap (0, 1).$$

Let E be either $E^{\alpha}(f)$ or $E_{\alpha}(f)$. Noting that $f|((0, 1) - K)$ is continuous and $f|((0, 1) \cap K)$ is non-decreasing, we have both $E \cap ((0, 1) - K)$ and $E \cap ((0, 1) \cap K)$ in $F_{\sigma} \cap G_{\delta}$. Also, $R - (0, 1)$ is either contained in or disjoint from E. Hence $E \in F_{\sigma} \cap G_{\delta}$. Moreover, K is nowhere dense and $f|(R - K)$ is continuous. We see that the condition (*) is fufilled. That is, $E \in SM_2$. However, $f|K$ is discontinuous at every b_n and $\{b_n : n = 1, 2, \cdots\}$ is dense in K. There is no portion of K on which the restriction of f is continuous. Hence $f \notin \mathcal{DB}_1^*$.

Zahorski [5] defined a nested sequence of classes of functions $\mathcal{M}_i$ (i = 0, 1, $\cdots$, 5) and proved that $\mathcal{M}_0 = \mathcal{M}_1 = \mathcal{DB}_1$. We now check if $\mathcal{DB}_1^*$ fits somewhere in this sequence.

Theorem 3. (i) $\mathcal{DB}_1^* \subsetneqq \mathcal{M}_2$. (ii) There is no inclusion relation between $\mathcal{D}\mathcal{B}_1^*$ and $\mathcal{M}_i$ for $i = 3, 4, 5$.

Proof. (i) follows immediately from Lemma and the above example. For (ii), since $\mathcal{M}_3 \supset \mathcal{M}_4 \supset \mathcal{M}_5$, it suffices to show that there exist $\varphi \in \mathcal{M}_5 - \mathcal{D}\mathcal{B}_1^*$ and $\psi \in \mathcal{D}\mathcal{B}_1^* - \mathcal{M}_3$.

Let $G \in G_\delta$ contain all rational numbers and have Lebesgue measure zero. By Lemma 11 in [5], there exists $\varphi : R \to [0, 1]$ in $\mathcal{M}_5$ such that $\{x : \varphi(x) = 0\} = G$. A moment's reflection shows that $\varphi \notin \mathcal{B}_1^*$.

Now we define ψ as follows:

$$\psi(x) = \frac{1}{2} \quad \text{if } x \leq 0,$$

$$= 0 \quad \text{if } x \geq 1 \text{ or } x = \frac{1}{2^n}, \ n = 1, 2, \cdots,$$

$$= 1 \quad \text{if } x = \frac{1}{2}\left(\frac{1}{2^{2n}} + \frac{1}{2^{2n-1}}\right), \ n = 1, 2, \cdots,$$

$$= -1 \quad \text{if } x = \frac{1}{2}\left(\frac{1}{2^{2n+1}} + \frac{1}{2^{2n}}\right), \ n = 0, 1, 2, \cdots,$$

and let ψ be linear on intervals $\left[\frac{1}{2^{n+1}}, \frac{1}{2}\left(\frac{1}{2^{n+1}} + \frac{1}{2^n}\right)\right]$ and $\left[\frac{1}{2}\left(\frac{1}{2^{n+1}} + \frac{1}{2^n}\right), \frac{1}{2^n}\right]$, $n = 0, 1, 2, \cdots$. Clearly $\psi \in \mathcal{D}$ and ψ is continuous on $R - \{0\}$. It follows that $\psi \in \mathcal{D}\mathcal{B}_1^*$. Let $E = \{x : \psi(x) > 0\}$, $x_0 = 0$ and $c > 1$. For $\varepsilon > 0$, there exists n such that $1/2^{2n} < \varepsilon$. Let $h = h_1 = 1/2^{2n+1}$. Then $h h_1 > 0$,

$h/h_1 < c$ and $h + h_1 < \varepsilon$. Since $[x_0 + h, x_0 + h + h_1] \cap E = \emptyset$, $\varphi \notin \mathcal{M}_3$. The theorem is proved.

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