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Janina Ewert, Department of Mathematics WSP,
76-200 Słupsk, Poland
J.S. Lipiński, Institute of Mathematics, University
of Gdańsk, 80-952 Gdańsk, Poland

ON POINTS OF CONTINUITY, QUASICONTINUITY AND CLIQUISHNESS OF REAL FUNCTIONS

Let X be a topological space and let Y be a metric space with metric d. By R^m we mean m-dimensional euclidean space.

A function $f : X \to Y$ is said to be:

- quasicontinuous at a point $x_0 \in X$, if for every neighborhood V of $f(x_0)$ and every neighborhood U of x_0 there exists a nonempty open set $U_1 \subset U$ such that $f(U_1) \subset V$ [1,3,5,6,8];
- cliquish at a point $x_0 \in X$, if for every $\varepsilon > 0$ and every neighborhood U of x_0 there exists a nonempty open set $U_1 \subset U$ such that $d(f(x'), f(x'')) < \varepsilon$ for $x', x'' \in U_1$ [4,5,6,8].

Let us denote by $C(f)$, $E(f)$ and $A(f)$ the set of points of continuity, quasicontinuity and cliquishness of f, respectively. Then we have $C(f) \subset E(f) \subset A(f)$, the set $A(f)$ is closed [4, Theorem 1] and $A(f) \setminus C(f)$ is of the first category [6, Theorem 7].

Let us consider a triplet C, E, A of subsets of X such that:

$$(*) \qquad \begin{cases} C \subset E \subset A = \overline{A}, \ C \text{ is } G_\delta \text{ and} \\ A \setminus C \text{ is of the first category.} \end{cases}$$

Does there exist a function $f : X \to Y$ for which $C = C(f)$, $E = E(f)$ and $A = A(f)$? When $X = Y = R^1$, a positive answer is given in [2, Theorem 2]. In [2] there can be found a characterization of the pairs $E(f)$, $A(f)$ and $C(f)$, $E(f)$ when X and Y are uniform spaces. In this paper we will characterize the triplet $C(f)$, $E(f)$, $A(f)$ for a function f defined on R^m.

Theorem 1

The sets $C,E,A \subset R^m$ satisfy (*) if and only if there exists a function $f : R^m \to R^1$ such that $C = C(f)$, $E = E(f)$ and $A = A(f)$.

Proof: The necessity follows from [6, Theorem 7] and from the results of [4].

Assume that (*) is satisfied. Since $A\setminus C$ is an F_σ set of the first category, there exists a sequence $\{F_n : n=0,1,\dots\}$ of closed nowhere dense sets such that $F_0 = \emptyset$, $F_n \subset F_{n+1}$ for $n=1,2,\dots$ and $A\setminus C = \bigcup_{n=1}^{\infty} F_n$. Let $d(p,F_n)$ be the distance of a point p from the set F_n. For each $n=1,2,\dots$ we define the function $f_n : R^m \to R^1$ by

$$f_n(p) = \begin{cases} 10^{-n}\sin[d(p,F_n)]^{-1} & (p \notin F_n), \\ 2\cdot 10^{-n} & (p \in F_n\setminus(E \cup F_{n-1})), \\ 0 & (p \in F_{n-1} \cup (F_n \cap E)). \end{cases}$$

The function f_n is continuous on the set $R^m\setminus F_n$ and for any point $p_0 \in F_n$ we have $\limsup\limits_{\substack{p\to p_0 \\ p\notin F_n}} f_n(p) = 10^{-n}$ and

$\liminf_{\substack{p \to p_0 \\ p \notin F_n}} f_n(p) = -10^{-n}$. Let us put $g = \sum_{n=1}^{\infty} f_n$. The uniform convergence of this series implies the continuity of g on the set $R^m \setminus \bigcup_{n=1}^{\infty} F_n = C \cup (R^m \setminus A)$. Let $s_{n-1} = \sum_{k=1}^{n-1} f_k$. Evidently $g = s_{n-1} + f_n + r_n$. If $p_0 \in F_n \setminus F_{n-1}$, then s_{n-1} is continuous at p_0 and $|r_n| \leq \frac{2}{9} \cdot 10^{-n}$, so we obtain the following inequalities

$$(1) \qquad s_{n-1}(p_0) + \frac{7}{9} \cdot 10^{-n} \leq \limsup_{\substack{p \to p_0 \\ p \notin F_n}} g(p) \leq \leq s_{n-1}(p_0) + \frac{11}{9} \cdot 10^{-n},$$

$$(2) \qquad s_{n-1}(p_0) - \frac{11}{9} \cdot 10^{-n} \leq \liminf_{\substack{p \to p_0 \\ p \notin F_n}} g(p) \leq \leq s_{n-1}(p_0) - \frac{7}{9} \cdot 10^{-n}.$$

Consequently

$$\frac{14}{9} \cdot 10^{-n} \leq \limsup_{\substack{p \to p_0 \\ p \notin F_n}} g(p) - \liminf_{\substack{p \to p_0 \\ p \notin F_n}} g(p) \leq \frac{22}{9} \cdot 10^{-n}.$$

Hence the function g is discontinuous at each point of the set $A \setminus C = \bigcup_{n=1}^{\infty} F_n$. Thus we obtain

$$(3) \qquad C(g) = C \cup (R^m \setminus A).$$

Let us observe that altering the values of f_n on the set F_n does not change (1), (2) or (3) provided $|f_n(p)| \leq 2 \cdot 10^{-n}$ for $p \in F_n$. Moreover the density of the set $C(g)$ implies $A(g) = R^m$. Now let $p_0 \in A \setminus E$. Then $p_0 \in F_n \setminus (F_{n-1} \cup E)$ for some n and, by the definition of the function g, $g(p_0) = s_{n-1}(p_0) + 2 \cdot 10^{-n} > s_{n-1}(p_0) + \frac{11}{9} \cdot 10^{-n}$. According to (1) there exists a neighborhood

G of p_0 such that $g(p) \leq s_{n-1}(p_0) + \frac{12}{9} \cdot 10^{-n} < g(p_0)$ for any $p \in G \setminus F_n$. Hence g is not quasicontinuous at p_0. Thus

$$A \setminus E \subset A(g) \setminus E(g).$$

Moreover (4) and (3) are true if we change the values of the function f_n on the set $F_n \cap E$ provided $|f_n(p)| \leq 2 \cdot 10^{-n}$ for $p \in F_n \cap E$.

Let $p \in E \setminus C$. Then there exists exactly one index n such that $p \in (F_n \cap E) \setminus F_{n-1}$. Since $C(g)$ is a dense set, there exists a sequence $\{p_k : k=1,2,\ldots\}$ of points belonging to $C(g)$ converging to p. The sequence $\{g(p_k) : k=1,2,\ldots\}$ is bounded, so it contains a convergent subsequence. Without loss of generality we may assume that $\{g(p_k) : k=1,2,\ldots\}$ is convergent. Now we define for $n=1,2,\ldots$ functions h_n letting

$$h_n(p) = \begin{cases} \lim_{k\to\infty} g(p_k) - s_{n-1}(p) & (p \in (F_n \cap E) \setminus F_{n-1}), \\ f_n(p) & (p \notin (F_n \cap E) \setminus F_{n-1}). \end{cases}$$

Let us put $h = \sum_{n=1}^{\infty} h_n$. Since $p_k \in R^m \setminus F_n$ for $k=1,2,\ldots$, (1) and (2) imply $|h_n(p)| \leq 2\ 10^{-n}$. Thus, according to earlier remarks $C(h) = C \cup (R^m \setminus A)$ and $A \setminus E \subset A(h) \setminus E(h)$. Furthermore for any point $p \in E \setminus C$ we have $h(p) = s_{n-1}(p) + h_n(p) = \lim_{k\to\infty} h(p_k)$. Hence p is a point of quasicontinuity of h. Consequently $E \setminus C \subset E(h) \setminus C(h)$. Thus $C(h) = C \cup (R^m \setminus A)$, $E(h) = E \cup (R^m \setminus A)$ and $A(h) = R^m$.

Let D be a dense border subset[1] of R^m. Finally we define a function $h' : R^m \to R^1$ by $h'(p) = d(p,A)$ for $p \in D$ and $h'(p) = 0$ for $p \notin D$. Evidently $A = C(h') = E(h') = A(h')$. Then for the function $f = h + h'$ we have $C(f) = C$, $E(f) = E$ and $A(f) = A$.

<u>Theorem 2</u>

Let X and Y be real normed spaces and let X be a Baire space. The sets $C, E, A \subset X$ satisfy $(*)$ if and only if there exists a function $f : X \to Y$ for which $C = C(f)$, $E = E(f)$ and $A = A(f)$.

Proof: The necessity follows from [4] and [6].

As in the proof of Theorem 1 we can show that there exists a function $f_1 : X \to R^1$ such that $C = C(f_1)$, $E = E(f_1)$ and $A = A(f_1)$. (The existence of a dense border subset $D \subset X$ which appears in the last part of the proof follows from the theorem of Sierpiński [7]). Let M be a one dimensional subspace of Y and let $i_M : M \to Y$ be the embedding of the subspace M in the space Y. By $T : R^1 \to M$ we denote the natural isomorphism. Then $f = i_M \circ T \circ f_1 : X \to Y$ is the function for which $C = C(f)$, $E = E(f)$ and $A = A(f)$.

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(1) Editorial Comment: A dense border set is a dense set whose complement is also dense.

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