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ON THREADING CONTINUOUS FUNCTIONS THROUGH COMPACT SETS

1. Introduction.

Suppose E is a compact set in $[0,1] \times [0,1]$. Under what circumstances is it true that for every point $(x_0,y_0) \in E$ there exists a function f continuous on $[0,1]$ such that $f(x_0) = y_0$ and such that the graph of f is contained in E? A moment's reflection reveals that the sets $E_x = \{y : (x,y) \in E\}$ must all be nonempty and must somehow connect with each other. More precisely, one can easily verify that, if $x \to x_0 \in [0,1]$, then the sets E_x must converge to E_{x_0} in the Hausdorff metric; i.e., the function E^* which maps x into E_x must be a continuous function from $[0,1]$ into the space of compact subsets of $[0,1]$ furnished with the Hausdorff metric. One might ask whether this condition is sufficient as well as necessary. In Example 2.1 we show that the condition does not suffice in general. In Section 3 we show it is sufficient if each of the sets E_x is nowhere dense in $[0,1]$. In Section 4 we study the general setting.

This problem had its origins in a question related to differentiation theory. We discuss this connection briefly in Section 5.

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2. Preliminaries.

It is convenient to set apart some definitions and notations which appear frequently in the sequel.

Let $\mathcal{S}$ denote the family of all closed subsets of $[0,1]$ furnished with the Hausdorff metric d defined by

$$d(S_1,S_2) = \inf\{\delta > 0 : S_1^{(\delta)} \supset S_2 \text{ and } S_2^{(\delta)} \supset S_1\}$$

where, for example, $S_1^{(\delta)} = \{y : |x-y| < \delta \text{ for some } x \in S_1\}$.

It is an exercise in real analysis to show that $(\mathcal{S},d)$ is a compact metric space. If $F \subset [0,1]$, we shall denote the complement of F relative to $[0,1]$ by F^c.

Let E be a nonempty compact subset of $[0,1] \times [0,1]$. For each $x \in [0,1]$, let $E_x = \{y : (x,y) \in E\}$. If the function E^* defined by $E^*(x) = E_x$ is a continuous function from $[0,1]$ to $(\mathcal{S},d)$, then we call E an <u>admissible</u> set.

It is easy to show that E must be admissible if the problem posed in the Introduction has a solution. To see that this condition does not suffice consider Example 2.1 below.

<u>2.1 Example</u>. E consists of those portions of the following lines which lie in the closed unit square: $y = 1$, $x = 0$ and $y = kx$ for $k = 0,1,2,\ldots$. It is geometrically clear (but tedious to check) that the map E^* is continuous. However, it can also be shown that there is no continuous real function f on $[0,1]$ with graph$(f) \subset E$ and $f(0) \in (0,1)$. Indeed, any neighborhood of $(0,y_0)$ which does not contain $(0,0)$ meets E in a

countable number of disjoint line segments with $(0,y_0)$ contained in a vertical segment.

3. <u>Nowhere Dense Sections</u>.

Let E be an arbitrary admissible set. Let $f(x) = \inf\{E_x\}$. It is easy to verify that f is continuous and that graph$(f) \subset E$. Thus each admissible set contains the graph of some continuous function on $[0,1]$. Such a continuous function f cannot, in general, be chosen to satisfy $f(x_0) = y_0$ for any particular $(x_0,y_0) \in E$ as Example 2.1 illustrated. The difficulty we encounter in that example is that continuity of the function E^* requires that the entire segment $[0,1]$ be contained in E_0. Such a situation cannot occur if E_x is nowhere dense for every $x \in [0,1]$.

In this section we show that if E is an admissible set and E_x is nowhere dense for each $x \in [0,1]$, then our problem has a solution. The proof is based on several lemmas which together provide a convergent sequence of functions, the limit of which is a solution to the problem. The main construction is developed in 3.5 below.

For the remainder of this section we assume E is an admissible set with E_x nowhere dense in $[0,1]$ for each $x \in [0,1]$.

3.1 <u>Lemma</u>. Let $\varepsilon > 0$, $(x_0,y_0) \in E$, $x_0 < 1$. There exists a function g and a positive number δ such that g is relatively continuous on $[x_0, x_0+\delta]$, graph$(g) \subset E$ and $0 \leq g(x_0) - y_0 < \varepsilon$.

Proof. Choose $b \in E_{x_0}^c \cap (y_0, y_0+\varepsilon)$ and define g by $g(x) = \sup\{y : y \in E_x, \ y < b\}$. Since E is admissible, there exists $\delta > 0$ such that $b \in E_x^c$ for all $x \in [x_0, x_0+\delta]$. Since E is closed, the graph of g is contained in E, and the continuity of g on $[x_0, x_0+\delta]$ follows from the continuity of E^*. Finally, the inequalities $0 \leqq g(x_0) - y_0 < \varepsilon$ follow from the inequalities $y_0 < b < y_0 + \varepsilon$.

The function we seek is a limit of a sequence of functions, called ε-functions, which possess certain desirable properties.

3.2 *Definition*. Let $\varepsilon > 0$. A function f defined on a half-open interval $[a,b) \subset [0,1]$ is called an ε-*function* if:

(1) $\operatorname{graph}(f) \subset E$,

(2) f is right continuous,

(3) for every $x \in [a,b)$ there is $\delta > 0$ such that $x + \delta \leqq b$ and f is continuous on $(x, x+\delta)$, and

(4) for every $x \in (a,b)$, $0 \leqq f(x) - \overline{\lim}_{s \to x^-} f(s) < \varepsilon$.

The lemma below shows that ε-functions possess a certain Darboux-like property.

3.3 *Lemma*. If f is an ε-function on $[a,b)$ and $c \in (a,b)$ with $f(a) > y > f(c)$ for some number y, then there exists $x \in (a,c)$ such that $f(x) = y$.

Proof: Let $x = \inf\{t \in [a,c] : f(t) \leqq y\}$. Then also $x = \sup\{t \in [a,x] : f(t) \geqq y\}$. Thus $f(x) \leqq y$ by (2) and $f(x) \geqq y$ by (4), completing the proof.

3.4 *Lemma*. Let g be an ε-function on $[a,b)$. Then $\lim_{s \to b^-} g(s)$ exists.

Proof. Let $U = \overline{\lim}_{s \to b^-} g(s)$ and $L = \underline{\lim}_{s \to b^-} g(s)$. If $L \neq U$, then 3.3 implies that the entire segment $[L,U]$ is contained in E_b (since $(x,f(x)) \in E$ for all x and E is closed) which contradicts the underlying assumption that each set E_x is nowhere dense.

3.5 *Lemma*. Let $(x_0,y_0) \in E$, $\varepsilon_1 > \varepsilon > 0$ and g an ε_1-function on $[x_0,1)$ with $0 \leqq g(x_0) - y_0 < \varepsilon_1$. Then there exists an ε-function f on $[x_0,1)$ such that $f \leqq g$ on $[x_0,1)$ and $0 \leqq f(x_0)-y_0 < \varepsilon$.

Proof. Let $\mathcal{G} = \{(f,x) : f$ is an ε-function on $[x_0,x)$ such that $f \leqq g$ on $[x_0,x)$ and $0 \leqq f(x_0) - y_0 < \varepsilon\}$. Define an ordering $\leqq$ on $\mathcal{G}$ by: $(f_1,x_1) \leqq (f_2,x_2)$ whenever $x_1 \leqq x_2$ and $f_1 = f_2$ on $[x_0,x_1)$. We show by Zorn's Lemma that $\mathcal{G}$ contains a maximal element (f,x) with $x = 1$.

Suppose that $\{(f_\lambda,x_\lambda)\}$ is a totally ordered subset of $\mathcal{G}$ Define $x_\infty = \sup x_\lambda$ and define g_∞ on $[x_0,x_\infty)$ by $g_\infty(t) = \lim f_\lambda(t)$. It is obvious that conditions (1)-(4) of 3.2 hold, so (g_∞,x_∞) is an upper bound for $\{(f_\lambda,x_\lambda)\}$.

We combine the proof that $\mathcal{G}$ is non-void with the proof that any maximal element of $\mathcal{G}$ must have the form $(f,1)$ by showing that for any point $(z_1,y_1) \in E \cap \{(z_1,s) : s \leqq g(x_1)\}$, there is a $\delta > 0$ and a relatively continuous function g_1 defined on $[z_1,z_1+\delta)$ with $0 \leqq g_1(z_1) - y_1 < \varepsilon$, $g_1(t) \leqq g(t)$ and $\text{graph}(g_1) \subset E$. Taking $(z_1,y_1) = (x_0,y_0)$ will show that $\mathcal{G}$ is non-void. If (x_∞,g_∞) is a maximal element of $\mathcal{G}$, then taking $(z_1,y_1) = (x_\infty, \lim_{t \to x_\infty^-} g_\infty(t))$, which exists by 3.4, we see that $x_\infty = 1$ (or we contradict maximality of (x_∞,g_∞)). To construct g_1 and δ we consider 2 cases.

Case 1: $g(z_1) = y_1$. Choose $\delta > 0$ such that g is relatively continuous on $[z_1, z_1+\delta]$. (Such a δ exists by (2) and (3) of 3.2.) Set $g_1 = g$ on $[z_1, z_1+\delta)$.

Case 2: $g(z_1) > y_1$. By 3.1 there exist $\bar{\delta}_1 > 0$ and a relatively continuous function $\bar{g}_1$ on $[z_1,z_1+\bar{\delta}_1)$ such that $\text{graph}(\bar{g}_1) \subset E$ and $0 \leqq \bar{g}_1(z_1) - y_1 \leqq \min(\varepsilon, g(z_1)-y_1)$. Since g is an ε_1-function on $[x_0,1)$, there is a $\delta_1 > 0$ such that g is relatively continuous on $[z_1,z_1+\delta_1]$. Set $\delta = \min(\delta_1,\bar{\delta}_1)$ and $g_1 = \min(\bar{g}_1,g)$ on $[z_1,z_1+\delta)$.

Thus $\mathcal{G}$ is non-void and by Zorn's Lemma it contains maximal element (f,x), where $x = 1$ by the previous paragraph. This proves the lemma.

3.6 Lemma. Let $(x_0, y_0) \in E$. There exists a continuous function g on $[x_0, 1]$ such that graph$(g) \subset E$ and $g(x_0) = y_0$.

Proof. Let $g_0(x) = \sup E_x$. Then g_0 is continuous and, in particular, is a 1-function on $[x_0,1)$. Repeated use of 3.5 gives rise to a sequence of functions g_n on $[x_0,1)$ such that for each n, g_n is a $\frac{1}{n}$-function, $g_{n+1} \leq g_n$, and $0 \leq g_n(x_0) - y_0 < \frac{1}{n}$. Let $g = \inf_n g_n$. Then $g(x_0) = y_0$ and graph $(g) \subset E$. Set $g(1) = \lim_{n\to\infty} \lim_{x\to 1^-} g_n(x)$, which exists by 3.4. It remains to show that g is continuous.

Suppose $U = \overline{\lim}_{t\to x^-} g(x) > \underline{\lim}_{t\to x^-} g(x) = L$ for some $x \in (x_0,1]$. We show that $(L,U) \subset E_x$. Let $y \in (L,U)$ and $\delta \in (0, \min(y-L, U-y))$. Choose $t_1 < t_2 < x$ such that $x-t_1 < \delta$ and $|g(t_1) - U| < \delta$ and $|g(t_2) - L| < \delta$. Now choose N so that $g_N(t_1) - U < \delta$ and $g_N(t_2) - L < \delta$. Then $g_N(t_1) > y > g_N(t_2)$. It follows from 3.3 that there exists $c \in (t_1,t_2)$ with $g_N(c) = y$. It follows that $(L,U) \subset E_x$, contradicting our assumption that E_x is nowhere dense.

Similar arguments show that $\lim_{t\to x^+} g(t)$ exist for all $x \in [x_0,1)$ and that these limits equal $g(x)$ everywhere on $[x_0,1)$.

3.7 Theorem. Let E be an admissible subset of $[0,1] \times [0,1]$. If each of the sets E_x $(0 \leq x \leq 1)$ is nowhere dense, then for every $(x_0,y_0) \subset E$, there exists a continuous function f on $[0,1]$ such that graph$(f) \subset E$ and $f(x_0) = y_0$.

Proof. The result follows directly from 3.6 and its "left hand analogue" on $[0,x_0]$.

4. *The General Case*.

Example 2.1 shows that the (possibly) intuitive notion that the existence of segments in E_x "can only help matters" is incorrect. As we saw in §3, the *absence* of segments guarantees that certain limits must exist: in essence, if a bounded continuous function on (a,b) is not uniformly continuous, then the closure of the graph of f contains a vertical interval; similarly, if a uniformly bounded decreasing sequence of continuous functions $\{f_n\}$ has a discontinuous limit, then the closure of the union of the graphs of the $\{f_n\}$ contains a vertical interval.

The main result of this section is that if E is admissible, then our problem has a solution if and only if to each $(x_0,y_0) \in E$ there corresponds an open interval I and a continuous function g on $I \cap [0,1]$ such that $\text{graph}(g) \subset E$, $x_0 \in I$ and $g(x_0) = y_0$; that is, E has the desired property locally. This will happen if E possesses this local property at all points belonging to vertical segments contained in E. It is easy to see that this will occur if, for example, each (one dimensional) interior point of such a vertical segment is a (two dimensional) interior point of E. That, in turn, happens if the function E^{c*} defined by $E^{c*}(x) = \overline{E_x^c}$ is continuous. Neither of these last two conditions is necessary, however, as Example 4.1 shows.

4.1 Example: $E_0 = [0,1]$, $E_{2^{-n}} = \{(2^{-n}, 2^{-n}k) : k = 0,\ldots,2^n-1\}$, E also contains the line segments joining the pairs

$$(2^{-n}, 2^{-n}k), \quad (2^{-n-1}, 2^{-n-1}(2k))$$

and

$$(2^{-n}, 2^{-n}k), \quad (2^{-n-1}, 2^{-n-1}(2k+1))$$

for all $n = 0,1,2,\ldots$ and $k = 0,\ldots,2^n-1$.
One can check that $x \to E_x$ is continuous while $x \to E_x^c$ is not continuous. Nor does $y \in E_x$ imply $(x,y) \in \mathrm{Interior}(E)$. Nonetheless, for each $(x_0,y_0) \in E$ there is clearly a continuous function f on $[0,1]$ with $\mathrm{graph}(f) \subset E$ and $f(x_0) = y_0$.

We begin with some notation and terminology.

A set $E \subset [0,1] \times [0,1]$ is said to have the local continuity property if E is a closed set with the property that for any $(x,y) \in E$ there is an open interval I containing x and a continuous real function f with domain $I \cap [0,1]$, $f(x) = y$ and $\mathrm{graph}(f) \subset E$. Let (x_0,y_0) denote a fixed element of E with $x_0 \neq 1$. We shall define a mostly continuous E-function from (x_0,y_0) to be a pair (f, Γ_f), where f is a real function on a non-degenerate interval $[x_0, t_f] \subset [x_0,1]$, $f(x_0) = y_0$, Γ_f is a closed subset of $[x_0,t_f]$ not containing x_0 which is well-ordered by the usual ordering, f is relatively continuous on $[x_0,t_f]\backslash\Gamma_f$, $\mathrm{graph}(f) \subset E$ and for every $t \in \Gamma_f$, $\frac{1}{2}(\limsup_{s \to t^-} f(s) + \liminf_{s \to t^-} f(s)) = f(t) = \lim_{s \to t^+} f(s)$ (where the last equality is ignored if $t = t_f$).

We define a partial ordering on the mostly continuous E-functions from (x_0,y_0) by $(f,\Gamma_f) > (g,\Gamma_g)$ if $t_f \geq t_g$, $f(x) = g(x)$ for every $x \leq t_g$, $\Gamma_f \supset \Gamma_g$ and if $s \in \Gamma_f$, $t \in \Gamma_g$ with $t \geq s$, then $s \in \Gamma_g$. An alternate form of the last condition is $\Gamma_f \supset \Gamma_g$ and $\Gamma_f \setminus \Gamma_g \subset (\sup(\Gamma_g), t_f]$.

4.2 Lemma. Let E be a subset of $[0,1] \times [0,1]$ which has the local continuity property and let $(x_0,y_0) \in E$ with $x_0 \neq 1$. There is a mostly continuous E-function from (x_0,y_0), denoted by (f,Γ_f), such that domain$(f) = [x_0,1]$.

Proof. Since E has the local continuity property, there exist mostly continuous E-functions from (x_0,y_0), and if (g,Γ_g) is a maximal mostly continuous E-function from (x_0,y_0), then domain$(g) = [x_0,1]$. Thus by Zorn's Lemma it suffices to show that any totally ordered family $\{(f_\lambda,\Gamma_{f_\lambda})\}$ of mostly continuous E-functions from (x_0,y_0) has an upper bound.

Let $t_f = \sup_\lambda\{t_{f_\lambda}\}$ and let $\Gamma_f = \bigcup_\lambda \Gamma_{f_\lambda} \cup \{t_f\}$.

Define f on $[x_0,t_f]$ as follows. If $x \in [x_0,t_f)$, then there is a λ_0 such that for all $\lambda \geq \lambda_0$, $f_\lambda(x) = f_{\lambda_0}(x)$. Thus $\lim_\lambda f_\lambda(x)$ exists, so set $f(x) = \lim_\lambda f_\lambda(x)$. Define

$$f(t_f) = \tfrac{1}{2}\Big(\varlimsup_{x \to t_f^-} f(x) + \varliminf_{x \to t_f^-} f(x)\Big).$$

It remains to show that $(t_f, f(t_f)) \in E$, for then it will be clear that (f,Γ_f) is a mostly continuous E-function from (x_0,y_0) which is an upper bound for $\{(f_\lambda,\Gamma_{f_\lambda})\}$.

To show that $(t_f, f(t_f)) \in E$ we consider two cases.

<u>Case 1</u>: f is left continuous at t_f.

Since $\lim_{t \to t_f^-}(t, f(t)) = (t_f, f(t_f))$, $(t, f(t)) \in E$ for all $t < t_f$ and E is closed, it follows that $(t_f, f(t_f)) \in E$.

<u>Case 2</u>: $U = \overline{\lim}_{t \to t_f^-} f(t) > L = \underline{\lim}_{t \to t_f^-} f(t)$.

Let $\delta > 0$. We shall show that there is a $c \in (t_f-\delta, t_f)$ such that $f(c) = f(t_f) = \frac{1}{2}(U+L)$. Choose $a,b \in (t_f-\delta, t_f)$ and λ such that $f = f_\lambda$ and $[x_0,b]$, $a < b$, $f(a) > f(t_f)$, $f(b) < f(t_f)$. Set $c = \inf\{t \in [a,b] : f(t) \leq f(t_f)\}$. Then $f(t) > f(t_f)$ for all $t \in [a,c)$. Since $f = f_\lambda$ on $[a,c]$ and $(f_\lambda,\Gamma_{f_\lambda})$ is a mostly continuous E-function from (x_0,y_0), we conclude that $f = f_\lambda$ is continuous at c. Thus $f(c) = f(t_f)$, so $(t_f, f(t_f)) \in E$, since $(c, f(c)) \in E$ and $\mathrm{dist}((c, f(c)), (t_f, f(t_f))) < \delta$.

We are now ready to establish the main result of this section - if E has the local continuity property, then continuous functions meeting specified initial conditions exist on $[0,1]$. The proof of the theorem follows natural lines but is rather lengthy because of a technical difficulty which arises. Because of this, we devote a paragraph to describing this difficulty as well as our manner of resolving it.

The basic idea behind the proof is to use the mostly continuous function f whose existence is guaranteed by 4.2 as a starting point. We modify f near points of the set Γ_f in order to construct the desired function g. The inductive process follows natural lines. The technical difficulty arises when we attempt to define g_β in terms of the functions g_α, $\alpha < \beta$. To create continuity at x_β requires "backtracking": a redefinition on some interval $[x_\beta - \delta, x_\beta]$, thus we cannot create the desirable extension property that each g_β extends all preceding g_α. How do we thus construct a single function g with all desired properties? The trick is to predetermine the magnitudes of these "backtrackings" in a manner which permits an adequate substitute for the unavailable desirable extension property. We do this by means of the function j appearing at the beginning of the proof of the theorem.

<u>4.3 Theorem</u>. Let E have the local continuity property and let $(x_o, y_o) \in E$. There is a continuous real function g on $[0,1]$ with $g(x_0) = y_0$ and $\mathrm{graph}(g) \subset E$.

<u>Proof</u>. By 4.2 let (f, Γ_f) be a mostly continuous E-function from (x_o, y_o) with domain $(f) = [x_o, 1]$, and let $\Gamma_f = \{x_\alpha\}_{1 \leq \alpha < \beta_1}$ be a listing indexed by the ordinals less than β_1 with the ordering preserved (i.e., $\alpha_1 > \alpha_2$ implies $x_{\alpha_1} > x_{\alpha_2}$). Now β_1 is countable since $\Gamma \subset [0,1]$ and $x_{\alpha+1} - x_\alpha > 0$ for each $1 \leq \alpha + 1 < \beta_1$. Thus we can define a 1-1 correspondence $\alpha \to N_\alpha$ between the ordinal interval $[1, \beta_1)$ and the positive integers.

Define $j: [1,\beta_1) \to [0,1]$ by

$$j(\alpha) = \begin{cases} 2^{-n_\alpha} & \text{if } \alpha \text{ is a limit ordinal,} \\ x_\alpha - x_{\alpha-1} & \text{if } \alpha-1 \text{ exists.} \end{cases}$$

If α is a limit ordinal and $\alpha < \beta_1$, we define $U_\alpha = \overline{\lim}_{x \to x_\alpha^-} f(x)$ and $L_\alpha = \underline{\lim}_{x \to x_\alpha^-} f(x)$. For any ordinal α with $\alpha + 1 < \beta_1$, we define, $U_{\alpha+1} = \sup\{f(x) : x \in [x_\alpha, x_{\alpha+1}]\}$ and $L_{\alpha+1} = \inf\{f(x) : x \in [x_\alpha, x_{\alpha+1}]\}$.

We now define by transfinite induction a family $\{g_\alpha\}_{1 \le \alpha < \beta_1}$ of functions with the following properties.

(1) For each α, g_α is relatively continuous on $[x_o, x_\alpha]$, $\text{graph}(g_\alpha) \subset E$, $g_\alpha(x_o) = y_o$, $g_\alpha(x_\alpha) = f(x_\alpha)$.

(2) If $1 \le \delta < \alpha$ are ordinals $< \beta_1$ with $x_\delta \le x_\alpha - j(\alpha)$, then $g_\alpha = g_\delta$ on $[x_o, x_\delta]$.

(3) For each α,

$$\text{range}(g_{\alpha+1} \mid [x_\alpha, x_{\alpha+1}]) \supset [\tfrac{3}{4} L_{\alpha+1} + \tfrac{1}{4} U_{\alpha+1}, \tfrac{1}{4} L_{\alpha+1} + \tfrac{3}{4} U_{\alpha+1}].$$

Suppose $\{g_\alpha\}$ have been defined for all $0 < \alpha < \beta$ to satisfy conditions (1) - (3). To define g_β we consider two cases.

Case 1: β is not a limit ordinal.

Subcase (a): f is continuous at x_β. Define $g_\beta = g_{\beta-1}$ on $[x_o, x_{\beta-1}]$ (if $\beta \ne 1$) and $g_\beta = f$ on $[x_{\beta-1}, x_\beta]$. It is easily checked that conditions (1) - (3) hold.

Subcase (b): $U = \overline{\lim}_{x \to x_\beta^-} f(x) > \underline{\lim}_{x \to x_\beta^-} f(x) = L$. Since E has the local continuity property, there is a $q > 0$ and a continuous function h defined on $[x_\beta - q, x_\beta]$ such that $h(x_\beta) = f(x_\beta)$ and graph(h) $\subset E$. Since $f(x_\beta) = \frac{1}{2}(U + L)$, the relative continuity of h on $[x_\beta - q, x_\beta]$ and f on $[x_{\beta-1}, x_\beta)$ imply that there are points $r < s < t \in [x_{\beta-1}, x_\beta]$ such that $f(t) = h(t)$, $f(r) > \frac{3}{4} U_\beta + \frac{1}{4} L_\beta$ and $f(s) < \frac{1}{4} U_\beta + \frac{3}{4} L_\beta$. Define $g_\beta = g_{\beta-1}$ on $[x_o, x_{\beta-1}]$ (if $\beta \neq 1$), $g_\beta = f$ on $[x_{\beta-1}, t]$ and $g_\beta = h$ on $[t, x_\beta]$. The existence of r and s insure that condition (3) is satisfied, while conditions (1) and (2) are easily checked.

Case 2: β is a limit ordinal. Since Γ_f is closed, $x_\beta = \lim_{\alpha \to \beta^-} x_\alpha$. We now define a continuous function ϕ on $[x_o, x_\beta)$ as follows. First note that $\lim_{\alpha \to \beta^-} j(\alpha) = 0$. (This follows from $\sum_{\alpha<\beta_1} j(\alpha) < 2$.) Thus for any $x \in [x_o, x_\beta)$ there are ordinals $\alpha_1 < \alpha_2$ such that $x < x_{\alpha_1} < x_{\alpha_2} < x_\beta$ and $x_\alpha - j(x_\alpha) > x_{\alpha_1}$ for all $\alpha_2 \leq \alpha < \beta$. By condition (2) we have $g_\alpha = g_{\alpha_1}$ on $[x_o, x_{\alpha_1}]$. Thus we can set $\phi(x) = \lim_{\alpha \to \beta^-} g_\alpha(x)$ and conclude that ϕ is continuous at x (since $\phi = g_{\alpha_2}$ in a neighborhood of x) and $\phi(x) = g_\alpha(x)$ for all $\alpha_2 \leq \alpha < \beta_1$. In particular, for any $\delta < \beta$ there is an $\alpha_o \geq \delta+1$ with $\alpha_o < \beta$ and $g_\alpha = g_{\delta+1}$ on $[x_o, x_{\delta+1}]$ for all $\beta > \alpha \geq \alpha_o$ (by the above facts and condition (2)).

Before we define g_β we must establish two facts about the function ϕ, namely that $\overline{\lim}_{x \to x_\beta^-} \phi(x) \geq \frac{1}{4} L_\beta + \frac{3}{4} U_\beta$ and $\underline{\lim}_{x \to x_\beta^-} \phi(x) \leq \frac{3}{4} L_\beta + \frac{1}{4} U_\beta$. We give details for the first of these, and the second is similar. By condition (3) for each $\delta < \beta$

there is a point $t_\delta \in [x_\delta, x_{\delta+1}]$ such that $g_{\delta+1}(t_\delta) \geq \frac{3}{4} U_{\delta+1} + \frac{1}{4} L_{\delta+1}$. For any such δ by the previous paragraph there is an $\alpha_o > \delta$ such that $\alpha_o < \beta_1$ and $g_\alpha = g_{\delta+1}$ on $[x_o, x_{\delta+1}]$ for all $\beta > \alpha$, $\alpha \geq \alpha_o$. Thus $\phi(t_\delta) = \lim_{\alpha \to \beta^-} g_\alpha(t_\delta) = g_{\delta+1}(t_\delta)$, so

$$\begin{aligned}
\overline{\lim_{x \to x_\beta^-}} \phi(x) &\geq \overline{\lim_{\delta \to \beta^-}} \phi(t_\delta) = \overline{\lim_{\delta \to \beta^-}} g_{\delta+1}(t_\delta) \\
&\geq \overline{\lim_{\delta \to \beta^-}} (\tfrac{3}{4} U_{\delta+1} + \tfrac{1}{4} L_{\delta+1}) \geq \tfrac{3}{4} \overline{\lim_{\delta \to \beta^-}} U_{\delta+1} + \tfrac{1}{4} \underline{\lim_{\delta \to \beta^-}} L_{\delta+1} \\
&= \tfrac{3}{4} U_\beta + \tfrac{1}{4} L_\beta.
\end{aligned}$$

The inequality $\underline{\lim_{x \to x_\beta^-}} \phi(x) \leq \frac{1}{4} U_\beta + \frac{3}{4} L_\beta$ is derived in a similar way.

Now to define g_β we have two subcases.

Subcase (a): $\lim_{x \to x_\beta^-} \phi(x) = Q$ exists. By the previous paragraph we have $U_\beta = L_\beta = f(x_\beta) = Q$, so we can simply define g_β on $[x_o, x_\beta]$ to be the relatively continuous extension of ϕ to $[x_o, x_\beta]$ (i.e. $\phi(x_\beta) = Q$). Conditions (1) - (3) clearly hold.

Subcase (b): As we showed above, $f(x_\beta) = \frac{1}{2}(U_\beta + L_\beta)$ lies between $\overline{\lim_{x \to x_\beta^-}} \phi(x)$ and $\underline{\lim_{x \to x_\beta^-}} \phi(x)$, so just as in Case 1, Subcase (b), above we can find $q > 0$, a relatively continuous function h on $[x_\beta - q, x_\beta]$ and a point $t \in [x_\beta - q, x_\beta)$ such that $(x_\beta - t) < 2^{-n_\beta}$ and $h(t) = \phi(t)$. If $g_\beta = \phi$ on $[x_o, t]$ and $g_\beta = h$ on $[t, x_\beta]$, it is easily shown to satisfy conditions (1)-(3).

Since Γ_f is closed, β_1 is not a limit ordinal, so either $x_{\beta_1-1} = 1$, in which case g_{β_1-1} is the desired function g; or else f is continuous on $[x_{\beta_1-1}, 1]$, so

$$g(x) = \begin{cases} g_{\beta_1-1}(x) & \text{if } x_o \leq x \leq x_{\beta_1-1} \\ f(x) & \text{if } x_{\beta_1-1} < x \leq 1 \end{cases} \quad \text{will work.}$$

The theorem follows by the above argument and its "left hand analogue" on $[0, x_o]$.

We end this section with two observations and a problem.

1) The conditions E^* and E^{c*} continuous together imply a solution for our problem relative to E. The former condition is necessary but the latter is not. If each E_x is nowhere dense, then the latter condition is met trivially.

(2) Theorem 4.3 is more general than Theorem 3.7 but a proof that "If E^* is continuous and if each E_x is nowhere dense, then E has the local continuity property.", seems no easier or shorter than the proof of 3.7.

Under the hypotheses of Section 3, if to each $(x_0,y_0) \in E$ and $\varepsilon > 0$ there corresponds a neighborhood I of x_0 and a continuous function f on I such that either $f(x_0) = y_0$ and $\text{graph}(f) \subset E$ or $|f(x_0)-y_0| < \varepsilon$ and $\text{graph}(f) \subset E^c$, then E has the local continuity property.

The problem occurs when E and E^c are so badly entwined that neither situation occurs. Theorem 3.7 tells us that such entwinements do not occur when E_x is nowhere dense for every $x \in [0,1]$.

3) Perhaps a sufficient condition can be phrased in terms of local arc-wise connectedness. We have been unable to find such a condition. We mention that local arcwise connectedness is not a necessary condition (let E be a Cantor-set of horizontal line segments, e.g., $E = [0,1] \times C$, C = Cantor set). Nor is local-arcwise connectedness at a point sufficient for the local continuity property at that point. (A simple example shows this.) But we have been unable to determine whether local arcwise connectedness at <u>every</u> point of E implies the local continuity property.

5. Applications to Path Derivatives.

The classical theorem that the Dini derivates of a continuous function are in Baire class at most 2 has been extended in various ways. In an attempt to obtain a unified theory, Alijani [1] has been studying Path Derivatives [2]. A collection of sets $E = \{E_x : x \in R\}$ is called a system of paths if for every $x \in R$, x is a point of accumulation of E_x. The E-derivates of a function F at a point x are defined in terms of the extreme limits of the difference quotient of f as $t \to x$ through points in E_x; e.g., $\overline{F}'_E(x) = \overline{\lim}_{\substack{t \to x \\ t \in E_x}} \frac{F(t) - F(x)}{t - x}$. If one wishes to imitate certain classical proofs in this setting, certain structures on the system E are needed. In particular, if the function E^* is continuous, as in Sections 3 and 4, then a tractable theory develops. If one interprets E as a subset of the plane, rather than as a system of paths, the conclusions of Theorems 3.7 and 4.3 suffice. These conclusions do not follow from the continuity of E^* alone, of course, but Alijani has shown that a somewhat weaker conclusion suffices.

REFERENCES

[1] Alijani Koopei, Doctoral Research, Department of Mathematics, UCSB.

[2] A.M. Bruckner, R. O'Malley, B. Thomson, Path Derivatives: A unified view of certain generalized derivatives, to appear.

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