Real Analysis Exchange Vol 7 (1981-82)

Zygmunt Wójtowicz, Wyzsza Szkoła Pedagogiczna, Słupsk

THE STRUCTURE OF THE SETS $\{x: f(x) = h(x)\}$ FOR A TYPICAL CONTINUOUS FUNCTION f AND FOR A CLASS OF LIPSCHITZ FUNCTIONS h.

1. Introduction.

Let R denote the space of real numbers and let C denote the space of continuous functions $f: [0,1] \to R$ equipped with the uniform norm $\|f\| = \sup\{|f(x)| : 0 \le x \le 1\}$.

A subset A of C is said to be residual in C if its complement $C \setminus A$ is of the first category in C. If $f \in C$ and $\varepsilon > 0$, the open ball $\{g \in C: \|g - f\| < \varepsilon\}$ of C is denoted as usual by $B(f,\varepsilon)$.

An interval $I \subset [0,1]$ is said to be a rational interval if both of its endpoints are rational, and I will be called an open interval if it is open relative to $[0,1]$.

Let f be a given function in C and let $\lambda \in R$. For every $c \in R$, the set $\{x: f(x) = \lambda x + c\}$ is called a level of f in the direction λ. By a level of f we mean, in general, a level of f in some direction $\lambda \in R$.

Let $a_{f,\lambda} = \inf\{f(x) - \lambda x: 0 \le x \le 1\}$ and

$$b_{f,\lambda} = \sup\{f(x) - \lambda x: 0 \le x \le 1\}.$$

The levels of a function $f \in C$ are said to be normal in a direction $\lambda \in R$ if there exists a countable dense set $E_{f,\lambda}$ in $(a_{f,\lambda}, b_{f,\lambda})$ such that the level $\{x: f(x) = \lambda x + c\}$ of f in the direction λ is

a) a perfect set when $c \notin E_{f,\lambda} \cup \{a_{f,\lambda}, b_{f,\lambda}\}$,

b) a single point when $c = a_{f,\lambda}$ or $c = b_{f,\lambda}$, and

c) the union of a non-empty perfect set P with an isolated point $x \notin P$ when $c \in E_{f,\lambda}$ (P and x depending on f, λ and c).

It has been proved by A.M. Bruckner and K.M. Garg [1, Theorem 4.8] that there exists a residual set of functions f in C such that the levels of f are normal in all but a countable dense set of directions Λ_f in R, and in each direction $\lambda \in \Lambda_f$ the levels of f are normal except that there is a unique element c of $E_{f,\lambda} \cup \{a_{f,\lambda}, b_{f,\lambda}\}$ for which the level $\{x: f(x) = \lambda x + c\}$ contains two isolated points in place of one.

A family of functions $H \subset C$ is called a <u>2-parameter family</u> if for every pair of numbers $x_1, x_2 \in [0,1]$ $(x_1 \neq x_2)$ and for every pair of numbers $y_1, y_2 \in R$ there exists a unique $h \in H$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

In [2] Bruckner and Garg raised the following question. What conditions on H will guarantee that the analogue of the above theorem holds on replacing the family of straight lines $\{\lambda x + c\}$ by H?

In the present paper we show that the above question has an affirmative answer if the 2-parameter family H is almost uniformly Lipschitz. For the proof of this fact we use the methods of Bruckner and Garg [1].

In §2 we state some properties of a 2-parameter family H and in §3 we show that the above question has an affirmative answer

(Theorem 1) when H is almost uniformly Lipschitz. The proofs are to appear in [3].

2. Properties of a 2-parameter family.

Let H denote a 2-parameter family of continuous functions.

A function $h \in H$ for which $c = h(0)$ and $\lambda - h(1) - h(0)$ will be denoted by $h_{\lambda,c}$. The number λ is called the increase of the function $h_{\lambda,c}$. If $H_\lambda = \{h \in H: h(1) - h(0) = \lambda\}$, then it is clear that $H_{\lambda_1} \cap H_{\lambda_2} = \emptyset$ when $\lambda_1 \neq \lambda_2$ and $\bigcup_{\lambda \in R} H_\lambda = H$.

1. Proposition. If $x_0 \in [0,1]$, $y_0 \in R$ and the functions $h, h_1 \in H$, $h \neq h_1$ are such that $h(x_0) = h_1(x_0) = y_0$, then either $h(x) < h_1(x)$ when $0 \leq x < x_0$ and $h(x) > h_1(x)$ when $x_0 < x \leq 1$; or $h(x) > h_1(x)$ when $0 \leq x < x_0$ and $h(x) < h_1(x)$ when $x_0 < x \leq 1$.

2. Proposition. For every triple of numbers $x_0 \in [0,1]$ and y_0, $\lambda \in R$ there exists a unique function $h \in H_\lambda$ such that $h(x_0) = y_0$.

3. Proposition. $\lim_{n\to\infty} \|h_{\lambda_n,c_n} - h_{\lambda,c}\| = 0$ if and only if $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} \lambda_n = \lambda$.

4. Proposition. For every natural number n, let (x_n', y_n'), (x_n'', y_n''), (x', y'), $(x'', y'') \in [0,1] \times R$ ($x_n' \neq x_n''$ and $x' \neq x''$) and let $h_{\lambda_n, c_n}, h_{\lambda,c} \in H$ be functions such that $h_{\lambda_n,c_n}(x_n') = y_n'$,

$h_{\lambda_n, c_n}(x_n'') = y_n''$, $h_{\lambda,c}(x') = y'$ and $h_{\lambda,c}(x'') = y''$.

Then if $\lim_{n\to\infty} (x_n', y_n') = (x', y')$ and $\lim_{n\to\infty} (x_n'', y_n'') = (x'', y'')$,

then $\lim_{n\to\infty} \|h_{\lambda_n, c_n} - h_{\lambda,c}\| = 0$.

3. The structure of the set $\{x: f(x) = h(x)\}$ when $f \in C$ and h belongs to a 2-parameter family of continuous functions that is almost uniformly Lipschitz.

Let $f \in C$, $h \in H$ and let I be a subinterval of $[0,1]$. The graph of h is said to support the graph of f in I from above(below), if $h(x) \geq f(x)$ $(h(x) \leq f(x))$ for every $x \in I$ and there exists a point x_0 in I such that $h(x_0) = f(x_0)$. Further, if the point x_0 is not unique, then the graph of h is said to support the graph of f in I from above (below) at more than one point. We will say that the graph of h supports the graph of f in I if the graph of h supports the graph of f from above or from below.

If I and J are two disjoint subintervals of $[0,1]$, the graph of h will be said to support the graph of f in I and J, if it supports the graph of f in I as well as the graph of f in J. Similarly for three or more mutually disjoint subintervals of $[0,1]$.

1. Lemma. For every function $f \in C$ there is at most a countable set of functions in H which support the graph of f in two (or more) disjoint open subintervals of $[0,1]$.

Let $f \in C$ and $\lambda \in R$. We denote by

$$\alpha_{f,\lambda} = \inf \{c \in R: \{x: f(x) = h_{\lambda,c}(x)\} \neq \emptyset\}$$

$$\beta_{f,\lambda} = \sup \{c \in R: \{x: f(x) = h_{\lambda,c}(x)\} \neq \emptyset\}.$$

2. Lemma. For every function $f \in C$ and for every number $\lambda \in R$ the graph of the functions $h_{\lambda,\alpha_{f,\lambda}}$ and $h_{\lambda,\beta_{f,\lambda}}$ support the graph of f in $[0,1]$ from above and from below respectively at least at one point.

3. Lemma. For every function $f \in C$ there is at most a countable set $\Lambda_f \subset R$ such that for every $\lambda \in R \setminus \Lambda_f$

a) the sets $\{x: f(x) = h_{\lambda,\alpha_{f,\lambda}}(x)\}$ and $\{x: f(x) = h_{\lambda,\beta_{f,\lambda}}(x)\}$ consist of single points and

b) the set $E_{f,\lambda}$ of numbers c such that the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is not perfect is dense in $(\alpha_{f,\lambda}, \beta_{f,\lambda})$.

4. Lemma. There exists a residual set of functions f in C such that for every open rational interval $I \subset [0,1]$ the increases of functions in H of which the graphs support the graph of f in I from above at more than one point form a dense set in R and the increases of functions in H of which the graphs support the graph of f in I from below at more than one point form a dense set in R.

5. Lemma. The set of functions $f \in C$ of which the graphs support at least one function of H at more than two points is of

the first category in C.

6. Lemma. The set of functions $f \in C$ for which there exists $\lambda \in R$ and there exist two different functions $h_{\lambda,c_1}, h_{\lambda,c_2} \in H$ whose graphs support the graph of f in two different points is of the first category in C.

A 2-parameter family H of continuous functions is almost uniformly Lipschitz if

$$\forall_{c\in R} \forall_{\lambda\in R} \exists_{L_{\lambda,c}\geq 0} \forall_{x_1,x_2\in[0,1]} |h_{\lambda,c}(x_1) - h_{\lambda,c}(x_2)| \leq L_{\lambda,c}|x_1-x_2|$$

and, for every natural number n,

$$M_n = \sup \{L_{\lambda,c}: \lambda \in [-n,n],\ c \in [-n,n]\} < +\infty .$$

7. Lemma. Let H be a 2-parameter family of continuous functions which is almost uniformly Lipschitz.

Then there exists a residual set of functions $f \in C$ such that for every function $h \in H$ the function $f - h$ is not monotone at any point $x \in [0,1]$.

1. Theorem. Let H be a 2-parameter family of continuous functions which is almost uniformly Lipschitz.

Then there exists a residual set of functions $f \in C$ for which there exists a countable dense set $\Lambda_f \subset R$ and a set $E_{f,\lambda}$ countable and dense in $(\alpha_{f,\lambda}, \beta_{f,\lambda})$ such that

1^0 if $\lambda \in R\setminus\Lambda_f$, then

a) the sets $\{x: f(x) = h_{\lambda,\alpha_{f,\lambda}}(x)\}$ and $\{x: f(x) = h_{\lambda,\beta_{f,\lambda}}(x)\}$ consist of single points,

b) for $c \in (\alpha_{f,\lambda},\beta_{f,\lambda}) \setminus E_{f,\lambda}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is perfect and

c) for $c \in E_{f,\lambda}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is the union of a non-empty perfect set and an isolated point, and

2^0 if $\lambda \in \Lambda_f$, then

a) there exists an unique number $c_{f,\lambda} \in E_{f,\lambda} \cup \{\alpha_{f,\lambda},\beta_{f,\lambda}\}$ such that if $c_{f,\lambda} \in E_{f,\lambda}$, then $\{x: f(x) = h_{\lambda,c_{f,\lambda}}(x)\}$ is the union of a non-empty perfect set and two isolated points, and if $c_{f,\lambda} = \alpha_{f,\lambda}$ or $c_{f,\lambda} = \beta_{f,\lambda}$, then the set $\{x: f(x) = h_{\lambda,c_{f,\lambda}}(x)\}$ consists of two different points,

b) for $c \in E_{f,\lambda} \setminus \{c_{f,\lambda}\}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is the union of a non-empty perfect set and an isolated point,

c) for $c \in \{\alpha_{f,\lambda},\beta_{f,\lambda}\} \setminus \{c_{f,\lambda}\}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ consists of a single point and

d) for $c \in (\alpha_{f,\lambda},\beta_{f,\lambda}) \setminus E_{f,\lambda}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is perfect.

References

[1] A.M. Bruckner and K.M. Garg - The level structure of a residual set of continuous functions, Trans. Amer. Math. Soc. 232 (1977), p. 307-321.

[2] A.M. Bruckner and K.M. Garg - The level structure of typical continuous functions, Real Analysis Exchange Vol. 2 (1976) p. 35-39.

[3] Z. Wojtowicz - O typowych postaciach zbiorów {x: f(x) = h(x)} dla pewnych klas funkcji ciagłych, Słupskie Prace Matematyczno-Przyrodnicze (to appear).