INROADS *Real Analysis Exchange Vol. 7 (1981-82)*

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Continuity in the Density Topology

Andrew M. Bruckner in his book [2] (p. 21) states the following problem: characterize the set of all continuous f such that for each approximately continuous g, $g\circ f$ is also approximately continuous.

The obvious candidate for the solution to this problem is that f^{-1} preserves points of density of each set S in the range of f; that is, if x is a point of density of S, $f^{-1}(x)$ is a point of density of $f^{-1}(S)$. In his paper [1] Bruckner investigates this property for homeomorphisms.

In this paper the general class of functions will be investigated. This requires the concept of the density topology.

The following notation will be used:

R - the set of real numbers;

A^c - the complement of A;

J - any interval in R, open, closed or half-open;

$\mathcal{D}$ - the density topology on R (or on J);

$|A|$ - the outer Lebesgue measure of a set $A\subset R$;

$\bar{d}(A,x)$ $\underline{d}(A,x)$, $d(A,x)$ - the upper (resp. lower, ordinary) outer density of a set A at a point x;

$A^\wedge$ - the closure of a set $A\subset R$ relative to the density topology;

A° - the interior of a set $A\subset R$ relative to the density topology.

1.1. For a set $E \subset R$ let us denote by $d(E)$ the set of all $x \in R$ such that $d(E,x) = 1$. The class of all measurable sets E such that $E \subset d(E)$ is a topology for R, so called density topology $\mathcal{D}$ (see [6], p. 90).

A function $h: J \to R$ continuous as a mapping from J equipped with $\mathcal{D}$ topology into R with the same topology will be called continuous in the density topology or simply $\mathcal{D}$-continuous.

1.2. Proposition. _A function_ $h: J \to R$ _is approximately continuous if and only if it is continuous as a mapping from_ J _with_ $\mathcal{D}$ _topology into_ R _with the natural topology_.

Proof. See [2] p. 23.

1.3. Proposition. $\mathcal{D}$ _is a_ $T_{3\frac{1}{2}}$ _topology_.

Proof. See [3], see also [8] p. 26.

1.4. Proposition. _Let_ X _be a_ T_1 _topological space_. _It is_ $T_{3\frac{1}{2}}$ _if and only if a function_ $f: X \to X$ _is continuous whenever the composition_ $g \circ f$ _is continuous for every bounded continuous_ $g: X \to R$.

Proof. Necessity. Suppose X is $T_{3\frac{1}{2}}$ and there is a discontinuous function $f: X \to X$ such that for each real-valued bounded continuous g, $g \circ f$ is also continuous. Then there is a point $x \in X$ and a set $A \subset X$ such that $x \in A^-$ and $f(x) \notin f(A)^-$ (E^- is the closure of a set E in X). Since X is $T_{3\frac{1}{2}}$, there is a continuous function $g: X \to [0,1]$ such that $g(f(x)) = 0$ and $f(A)^- \subset g^{-1}(\{1\})$. Let $k = (g \circ f)^{-1}(\{1\})$. K is closed and $K \supset f^{-1}(f(A)^-) \supset A$. Since $x \in A^-$, then $x \in K$. Hence $g(f(X)) = 1$, a contradiction.

Sufficiency. Since X is $T_{3\frac{1}{2}}$, the subbase for X is formed by the sets $g^{-1}(G)$, where G is open in R, and g is a bounded continuous real-valued function on X (see [7] p. 96). Thus a function $f:X\to X$ is continuous if and only if $f^{-1}(g^{-1}(G)) = (g\circ f)^{-1}(G)$ is open for any G and g.

1.5. Corollary. A function $f:J\to J$ is $\mathcal{D}$-continuous if and only if the composition $g\circ f$ is approximately continuous for every continuous g.

Proof. This is a simple consequence of 1.2., 1.3. and 1.4..

2.1. Observation. If $A\subset R$ is measurable, then $A^\circ = A\cap d(A)$ and $A^\wedge = A\cup \{x\in R:\bar{d}(A,x) > 0\}$.

2.2. Proposition. For every $A\subset R$

$$A^\wedge = A\cup\{x\in R:\bar{d}(A,x) > 0\}.$$

Proof. It suffices to consider a nonmeasurable A. Let B be a measurable set such that $A\subset B$ and $|A\cap P| = |B\cap P|$ for every interval $P\subset R$. The inclusion $A\cup\{x\in R:\bar{d}(A,x) > 0\}\subset A^\wedge$ is obvious.

To prove the inclusion $A^\wedge\subset A\cup\{x\in R:\bar{d}(A,x) > 0\}$ suppose there is a point $x\in A^\wedge - A$ such that $\bar{d}(A,x) = 0$. Then $\bar{d}(B,x) = 0$. Hence $x\in(B^c\cup\{x\})^\circ$. Since $x\in A^\wedge$, then $(B^c\cup\{x\})\cap A \neq \emptyset$ and $x\in A$, a contradiction.

Note: x is an accumulation point of A for $\mathcal{D}$ topology if and only if $\bar{d}(A,x) > 0$.

2.3. Definition. A point $x\in R$ is called a point of inner density of a set $A\subset R$, if $d(A^c,x) = 0$. The set of all points of inner density of A

will be denoted by $d_i(A)$.

It is easy to see that x is a point of inner density of A if and only if there is a measurable set $E \subset A$ such that $x \in d(E)$. From 2.2 we also have:

2.4. Corollary. $A^\circ = d_i(A) \cap A$ _for every_ $A \subset R$.

3.1. Observation. _Each $\mathcal{D}$-continuous function is approximately continuous. Each $\mathcal{D}$-continuous function belongs to the first Baire class and has the Darboux property._

3.2. A set $A \subset R$ is measurable if and only if it is a union of a $\mathcal{D}$-open set and a $\mathcal{D}$-closed set. Thus we have:

Proposition. _If_ f _is $\mathcal{D}$-continuous, then_ $f^{-1}(B)$ _is measurable for every measurable set_ B.

The converse is false. It will be shown later.

3.3. Proposition. _A function_ f _is $\mathcal{D}$-continuous if and only if for each_ $B \subset R$ _and_ $y \in B \cap di(B)$,

$$f^{-1}(\{y\}) \subset d_i(f^{-1}(B)).$$

Proof. Simply, f is $\mathcal{D}$-continuous if and only if

$$f^{-1}(B^\circ) \subset (f^{-1}(B))^\circ.$$

3.4. Proposition. _A function_ $f:J \to R$ _is $\mathcal{D}$-continuous if and only if for each_ $A \subset J$ _and_ $x \in J$ _such that_ $\bar{d}(A,x) > 0$,

$$f(x) \in f(A) \quad \text{or} \quad \bar{d}(f(A), f(x)) > 0.$$

Proof. In fact, f is $\mathcal{D}$-continuous if and only if

$$f(A^{\wedge}) \subset f(A)^{\wedge}.$$

3.5. Proposition. _If_ f _satisfies the conditions_:

(i) $f^{-1}(B)$ _is measurable whenever_ B _is_;

(ii) _for every measurable_ B _and_ $y \in d(B) \cap B$,

$$f^{-1}(\{y\}) \subset f^{-1}(B)$$

and every $x \in f^{-1}(\{y\})$ _is a density point of_ $f^{-1}(B)$; _then_ f _is_ $\mathcal{D}$_-continuous_.

Proof. If f satisfies (i) and (ii), then $f^{-1}(B)$ is $\mathcal{D}$-open for every $\mathcal{D}$-open B.

4.1. Definition. A function $f: J \to R$ will be said _to preserve upper outer density_ at a point x, provided that for every set E,

$$\text{if} \quad \bar{d}(E, x) > 0, \quad \text{then} \quad \bar{d}(f(E), f(x)) > 0.$$

We say that f preserves upper outer density on a set $K \subset J$, if f preserves upper outer density at each point of K; f preserves outer density, if it does so on J.

4.2 Observation. _If_ $f: J \to R$ _preserves upper outer density_, _then it is_ $\mathcal{D}$_-continuous_.

4.3. Proposition. _Let_ $f: J \to R$ _be_ $\mathcal{D}$_-continuous and_ $x \in J$. _Then_ f _preserves upper outer density at_ x _if and only if_

$$d(f^{-1}(\{f(x)\}),x) = 0.$$

Proof. Necessity. Suppose

$$\bar{d}(f^{-1}(\{f(x)\}),x) > 0.$$

Let $E = f^{-1}(\{f(x)\})$. Then $\bar{d}(E,x) > 0$, but

$$\bar{d}(f(E),f(x)) = \bar{d}(\{f(x)\},f(x)) = 0.$$

Sufficiency. Let

$$d(f^{-1}(\{f(x)\}),x) = 0$$

and $\bar{d}(A,x) > 0$. Then also

$$\bar{d}(A - f^{-1}(\{f(x)\}),x) > 0.$$

Since f is $\mathcal{D}$-continuous and

$$f(x) \notin f(A - f^{-1}(\{f(x)\}))$$

then

$$\bar{d}(f(A - f^{-1}(\{f(x)\})),f(x)) > 0.$$

Hence

$$\bar{d}(f(A),f(x)) > 0.$$

4.4. Corollary. _A 1-1 function is $\mathcal{D}$-continuous if and only if it preserves upper outer density._

4.5. A 1-1 $\mathcal{D}$-continuous function $f:[a,b] \to R$ is a homeomorphism since it possesses the Darboux property. It is easy to see that a homeomorphism $f:[a,b] \to R$ preserves upper outer density if and only if f^{-1} preserves density points. Thus, for a homeomorphism, the conditions given in [1] and in this paper, are equivalent.

4.6. Proposition. _If_ $f:[a,b] \to R$ _is $\mathcal{D}$-continuous and 1-1_, _then_ f^{-1} _is absolutely continuous_.

Proof. As mentioned above, f is a homeomorphism. Thus f^{-1} is continuous and monotone. Moreover, f^{-1} transforms measurable sets into measurable sets (3.2); hence it fulfills the Lusin (N)-condition. The proposition now follows from the Banach-Zarecki theorem (see [4] p. 250).

4.7. Proposition. _If_ $f:[a,b] \to R$ _is $\mathcal{D}$-continuous and 1-1_, _then_ f _is absolutely continuous_.

Proof. It suffices to show that f satisfies the Lusin (N)-condition. Suppose there is a set Z of measure zero such that $|f(Z)| > 0$. Then there is a set K of type G_δ, of measure zero and such that $Z \subset K$. Also $|f(K)| > 0$. Since f is a homeomorphism $f(K)$ is of type G_δ. By the Lebesgue Density Theorem there exists a point $y \in f(K)$ which is a density point of $f(K)$. This y satisfies the condition

$$d(f([a,b] - K), y) = 0.$$

But each point $x \in [a,b]$ is a density point of $[a,b] - K$. Thus f does not preserve the upper outer density, a contradiction.

5.1. There exists a continuous function f which is not $\mathcal{D}$-continuous and such that $f^{-1}(B)$ is measurable whenever B is measurable.

Let (a_n) and (b_n) be sequences of real numbers converging to zero and satisfying the conditions:

(i) $b_1 > a_1 > b_2 > a_2 > \ldots > 0$;

(ii) $\bar{d}(\bigcup_{n=1}^{+\infty}(a_n,b_n),0) > 0$.

Let

$$f(x) = \begin{cases} 1/n & \text{for } x\in(a_n,b_n),\ n=1,2,\ldots \\ 0 & \text{for } x = 0 \end{cases}$$

and let f be linear in intervals $[a_{n+1},b_n]$, $n=1,2,\ldots$. It is easy to see that f is continuous and $f^{-1}(B)$ is measurable for any measurable B.

Also

$$\bar{d}(\bigcup_{n=1}^{+\infty} (a_n,b_n),0) > 0$$

but

$$0\notin f(\bigcup_{n=1}^{+\infty}(a_n,b_n)) \quad \text{and} \quad \bar{d}(f(\bigcup_{n=1}^{+\infty}(a_n,b_n)),0) = \bar{d}(\bigcup_{n=1}^{+\infty}\{\tfrac{1}{n}\},0) = 0,$$

so f is not $\mathcal{D}$-continuous.

5.2. There exists a function which is $\mathcal{D}$-continuous and not continuous. To give an example consider a sequence of intervals $((a_n,b_n))$ such that

(i) $a_n \to 0$, $b_n \to 0$;

(ii) $a_1 > b_1 > a_2 > b_2 > \dots > 0$;

(iii) $d(\bigcup_{n=1}^{+\infty}(a_n, b_n), 0) = 0$.

Let

$$f(x) = \begin{cases} 0 & \text{for } [0, b_1) \setminus \bigcup_{n=1}^{+\infty}(a_n, b_n) \\ 1 & \text{for } x = \frac{1}{2}(a_n + b_n),\ n=1,2,\dots \end{cases}$$

and let f be linear in intervals $[\frac{1}{2}(a_n + b_n), b_n]$, $[a_n, \frac{1}{2}(a_n + b_n)]$, $n=1,2,3,\dots$. Then $f: [0, b_1) \to R$. It is not continuous, since

$$\lim_{n\to\infty} f(\tfrac{1}{2}(a_n + b_n)) = 1.$$

But f is $\mathcal{D}$-continuous. It suffices to show its $\mathcal{D}$-continuity at the origin. Suppose $\bar{d}(A, 0) > 0$ and $f(0) = 0 \notin f(A)$. Then $A \subset \bigcup_{n=1}^{+\infty}(a_n, b_n)$, a contradiction with (iii).

5.3. There exists an approximately continuous function which is neither continuous nor $\mathcal{D}$-continuous.

Let (a_n) be a sequence of real numbers such that

(i) $a_1 > a_2 > \dots > 0$;

(ii) $\lim_{n\to\infty} a_n = 0$.

Let $T_n = (b_n, c_n)$, $n=1,2,\dots$ be such intervals that $[b_n, c_n] \subset (a_{n+1}, a_n)$, $n=1,2,\dots$ and

$$\bar{d}(\bigcup_{n=1}^{+\infty} T_n, 0) > 0.$$

Furthermore, let $J_n = (d_n, e_n)$, $n=1,2,\dots$ be such that $[d_n, e_n] \subset T_n$ and

$$d\left(\bigcup_{n=1}^{+\infty} J_n, 0\right) = 0.$$

Define a function f in the following way:

$$f(x) = \begin{cases} 0 & \text{for } x = a_n,\ n=1,2,3,\dots, \text{ or } x = 0 \\ n^{-1} & \text{for } x \in T_n - J_n,\ n=1,2,3,\dots \\ 1 & \text{for } x = \frac{1}{2}(d_n + e_n),\ n=1,2,3,\dots \end{cases}$$

and let f be linear in the intervals $[a_{n+}, b_n]$, $[d_n, \frac{1}{2}(d_n + e_n)]$, $[\frac{1}{2}(d_n + e_n), e_n]$, $[c_n, a_n]$, $n=1,2,3,\dots$. We have $f:[0,a\ ) \to \mathbb{R}$. It is easily seen that f is not continuous at the origin, since

$$\lim_{n\to\infty} f\left(\frac{1}{2}(d_n + e_n)\right) = 1.$$

As well, f is not $\mathfrak{D}$-continuous. We have

$$d\left(\bigcup_{n=1}^{+\infty} (T_n - J_n), 0\right) > 0$$

but $0 \notin f\left(\bigcup_{n=1}^{+\infty} (T_n - J_n)\right)$ and the set $f\left(\bigcup_{n=1}^{+\infty} (T_n - J_n)\right)$ is countable, hence its density at the origin is zero.

But f is approximately continuous. It suffices to show its approximate continuity at zero. Let $E = [0, a_1) - \bigcup_{n=1}^{+\infty} [d_n, e_n]$. Then $f|_E$ is continuous at 0 and 0 is a point of (right-hand) density of E. Thus f is approximately continuous.

5.4. Consider the set X of all homeomorphisms

$$h:[0,1] \xrightarrow[\text{onto}]{} [0,1].$$

It is a complete metric space with a metric

$$\rho(h_1,h_2) = \sup_{x\in[0,1]} |h_1(x) - h_2(x)| + \sup_{y\in[0,1]} |h_1^{-1}(y) - h_2^{-1}(y)|$$

(see [6] p. 50). The set Y of all $\mathcal{D}$-continuous 1-1 functions $f:[0,1] \xrightarrow[\text{onto}]{} [0,1]$ is included in X. As mentioned in 5.1., $X-Y$ is non-empty.

Proposition. Y <u>is of first category in</u> X.

Proof. The set $A = \{h\in X: h'(t) = 0 \text{ a.e.}\}$ is of the second category in X(*). If $h\in A$, then h is not absolutely continuous. By 4.7. we have $Y\subset X\setminus A$.

Note: There are absolutely continuous $h\in X$ such that $h\notin Y$. In [5] there is an example of a homeomorphism h such that h is $\mathcal{D}$-continuous while h^{-1} is not.

(*) This observation is due to Władysław Wilczyński.

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Received April 29, 1981 and in revised form February 19, 1982