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M. Laczkovich, Department I of Analysis, Eötvös Loránd University, Budapest, Muzeum krt. 6-8, H-1088, Hungary

Derivations on Differentiable Functions

Let $\mathbb{R}^I$ denote the class of real valued functions defined on the non-degenerate interval $I \subset \mathbb{R}$ and let $\mathcal{F} \subset \mathbb{R}^I$ be an algebra over the reals. A map

$$d : \mathcal{F} \to \mathbb{R}^I$$

is said to be a derivation if d is linear and

$$(1) \qquad d(fg) = f \cdot dg + g \cdot df$$

holds for every $f,g \in \mathcal{F}$. Suppose that the identity function $\sigma(x) = x$ $(x \in I)$ belongs to $\mathcal{F}$ and let $h = d\sigma$. Then $\mathcal{F}$ contains the polynomials and it is easy to see that $dp = h \cdot p'$ holds for every polynomials p. It was proved by Yasuo Watatani that $df = h \cdot f'$ holds also for every $f \in C^\infty(I)$ supposing that $C^\infty(I) \subset \mathcal{F}$ [2].

In this paper we describe the derivations of the class $\mathcal{D}$ of differentiable functions defined on I. As we shall see, $df = h \cdot f'$ is no longer true for every derivation on $\mathcal{D}$. However, if we suppose that df is Baire 1 for every $f \in \mathcal{D}$ then $(df)(x) = h(x) \cdot f'(x)$ holds true for each $f \in \mathcal{D}$ apart from a fixed scattered subset of I.

Theorem 1. A linear map

$$d: \mathcal{D} \to \mathbb{R}^I$$

is a derivation if and only if $(df)(a) = 0$ holds whenever $f \in \mathcal{D}$, $a \in I$ and

$$\lim_{x \to a} \frac{f(x)}{(x-a)^2} = 0 . \tag{2}$$

Proof. If (2) holds then the function g defined by

$$g(x) = \begin{cases} \frac{f(x)}{x-a} , & x \in I, \quad x \neq a \\ 0 , & x = a \end{cases}$$

is differentiable. Thus, if d is a derivation, we have

$$df = d(g \cdot (\sigma - a)) = dg \cdot (\sigma - a) + g \cdot h$$

from which $(df)(a) = 0$.

Now suppose that $d: \mathcal{D} \to \mathbb{R}^I$ is a linear map satisfying the condition of Theorem 1. Let $f, g \in \mathcal{D}$ and $a \in I$ be arbitrary and put

$$s(x) = (f(x) - f(a))(g(x) - g(a)) - f'(a) \cdot g'(a) \cdot (x-a)^2 \quad (x \in$$

Then $s \in \mathcal{D}$ and $\lim_{x \to a} \frac{s(x)}{(x-a)^2} = 0$ holds and hence we have

$$(ds)(a) = 0 .$$

Hence, by the linearity of d ,

$$d(fg)(a) = d(s+f(a)(g-g(a)) + g(a)(f-f(a)) + $$
$$+ f(a)g(a)+f'(a)g'(a)(\sigma-a)^2)(a) =$$
$$= (ds)(a) + f(a)(dg)(a) + g(a)(df)(a) +$$
$$+ 2 f'(a)\cdot g'(a)\cdot h(a)(a-a) = f(a)\cdot(dg)(a) +$$
$$+ g(a)\cdot(df)(a) .$$

Since a is arbitrary, this implies (1) and d is a derivation. □

The function f is said to have the second Peano derivative at the point a if the finite limit

$$(3) \qquad \lim_{x\to a} (f(x)-f(a)-f'(a)(x-a))/(x-a)^2$$

exists. We shall denote by $\mathcal{P}_2(a)$ the class of functions $f \in \mathcal{D}$ having the second Peano derivative at $a \in I$. Then $\mathcal{P}_2(a)$ is a linear subspace of $\mathcal{D}$ and $\mathcal{P}_2(a) \subsetneqq \mathcal{D}$ since the function $f(x) = (x-a)^2 \cdot \sin \frac{1}{x-a}$, $f(a) = 0$ belongs to $\mathcal{D} - \mathcal{P}_2(a)$.

<u>Corollary</u>. Let $d:\mathcal{D} \to \mathbb{R}^I$ be an arbitrary map and put $h = d\sigma$. d is a derivation if and only if for each $a \in I$ there exists a linear functional $\lambda_a:\mathcal{D} \to \mathbb{R}$ such that

$$\mathcal{P}_2(a) \subset \text{Ker } \lambda_a$$

and

$$(df)(a) = h(a)\cdot f'(a) + \lambda_a f$$

holds for every $f \in \mathcal{D}$ and $a \in I$.

Proof. Suppose that d satisfies the condition of the Corollary; then d is linear. Let $f \in \mathcal{D}$, $a \in I$ and suppose (2). Then

$$f(a) = f'(a) = 0 \quad \text{and} \quad f \in \mathcal{P}_2(a)$$

from which

$$(df)(a) = h(a)\cdot f'(a) + \lambda_a f = 0 \; .$$

Thus, by Theorem 1, d is a derivation.

On the other hand, if d is a derivation then let λ_a be defined by

$$\lambda_a f = (df)(a) - h(a)\cdot f'(a) \qquad (f \in \mathcal{D}) \; .$$

Then λ_a is linear on $\mathcal{D}$ and it easily follows from Theorem 1 that λ_a vanishes on $\mathcal{P}_2(a)$. In fact, let $f \in \mathcal{P}_2(a)$ and let c be the value of the limit under (3). Let $g = f - f(a) - f'(a)(\sigma - a) - c(\sigma - a)^2$, then $\lim_{x \to a} \frac{g(x)}{(x-a)^2} = 0$ and hence

$$\lambda_a f = \lambda_a g + \lambda_a\left(f(a) + f'(a)(\sigma - a) + c(\sigma - a)^2\right) =$$
$$= (dg)(a) - h(a)\cdot g'(a) + f'(a)\cdot h(a) - h(a)\cdot f'(a) = 0 \; . \; \square$$

Now we turn to formulate our main result. A set $H \subset \mathbb{R}$ is said to be scattered if H does not contain any non-empty subset which is dense in itself. It is obvious that every scattered set is nowhere dense in $\mathbb{R}$ and it is also well-known that a scattered set must be countable ([1], §18, V., p. 141).

Theorem 2. Let $d:\mathcal{D} \to \mathbb{R}^I$ a derivation and suppose that df is Baire 1 for every $f \in \mathcal{D}$. Then there exists a scattered set $H \subset I$ such that

$$(df)(x) = h(x) \cdot f'(x)$$

holds for every $f \in \mathcal{D}$ and $x \in I-H$. ($h = d\sigma$, where σ denotes the identity function on I.)

Lemma. Let $F:[0,\infty) \to \mathbb{R}$ be continuous, increasing and satisfying

$$(4) \qquad F(0) = F'_+(0) = 0 .$$

Let $g \in \mathcal{D}$, $a \in \operatorname{int} I$ be given and suppose that

$$|g(x)| \le F(|x-a|)$$

holds in a neighbourhood of a. Suppose further that a is an accumulation point of a given set $A \subset I$.

Then for every $\varepsilon > 0$ there are functions $g_1, g_2 \in \mathcal{D}$ and closed sets $K_1, K_2 \subset \mathbb{R}$ such that

(5) $\quad g_1(x) + g_2(x) = g(x)$ in a neighbourhood of a ;

(6) $|g_i(x)| \le \varepsilon$ for every $x \in I$ and $i=1,2$;

(7) $|g_i(x)| \le 4 \cdot F(\mathrm{dist}(x,K_i))$ for every $x \in I$ and $i=1,2$

(8) $\mathbb{R}-[a-\varepsilon,a+\varepsilon] \subset K_i$ $(i=1,2)$

and

(9) a is an accumulation point of $A \cap \mathrm{int}\, K_i$ $(i=1,2)$.

<u>Proof</u>. We can suppose $a=0$ and that 0 is a right hand side accumulation point of A. If $F(x) = 0$ for some $x > 0$ then $g(x) = 0$ in a neighbourhood of 0 and we can take $g_1 = g_2 = 0$ and $K_1 = K_2 = \mathbb{R}$. Thus we can suppose $F(x) > 0$ $(x > 0)$.

First we fix $\delta \in (0,\varepsilon)$ such that $[-\delta,\delta] \subset I$ and

(10) $|g(x)| \le F(|x|) < \frac{\varepsilon}{4}$ for every $|x| \le \delta$.

Let $b_o = \delta$. Suppose that $k \ge 0$ and the point $0 < b_k \le \delta$ has been defined. Since F is positive in $[\frac{1}{2}b_k,\delta]$ and uniformly continuous in $[0,\delta]$, we can choose $0 < c_k < \frac{1}{2}b_k$ such that

(11) $F(x) < 2 \cdot F(x-c_k)$ for every $x \in [\frac{1}{2}b_k,\delta]$.

Then we choose $0 < b_{k+1} < c_k$ with

(12) $A \cap (b_{k+1},c_k) \ne \emptyset$.

Thus, by induction, we have defined the sequences $\{b_k\}$, $\{c_k\}$ such that

$$\delta = b_o > c_o > b_1 > c_1 > \dots > 0$$

and

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} c_k = 0 .$$

Let $k \geq 0$ be fixed. By (10) and (11) we have

$$|g(\tfrac{1}{2}b_k)| \leq F(\tfrac{1}{2}b_k) < 2\cdot F(\tfrac{1}{2}b_k - c_k) .$$

Since F is continuous and positive for $x > 0$, we can construct a differentiable function $s_k:(c_k,\frac{1}{2}b_k] \to \mathbb{R}$ such that

(13) $$|s_k(x)| < 2\cdot F(x-c_k) \qquad (c_k < x \leq \tfrac{1}{2}b_k)$$

and

(14) $$s_k(\tfrac{1}{2}b_k) = g(\tfrac{1}{2}b_k) , \qquad s_k'(\tfrac{1}{2}b_k) = g'(\tfrac{1}{2}b_k) .$$

Now we define the functions g_1, g_2 as follows. For x negative we put $g_1(x) = g_2(x) = u(x)$, where

$$u(x) = \begin{cases} 0 & \text{if} \quad x \leq -\varepsilon \text{ and } x \in I , \\ \frac{1}{2}g(x) & \text{if} \quad -\frac{1}{2}\delta < x < 0 , \end{cases}$$

and in $(-\varepsilon,-\frac{1}{2}\delta]\cap I$ we define u such that it is differentiable, $u(-\frac{1}{2}\delta) = \frac{1}{2}g(-\frac{1}{2}\delta)$, $u'(-\frac{1}{2}\delta) = \frac{1}{2}g'(-\frac{1}{2}\delta)$ and satisfies the inequality

$$|u(x)| \le \min(\varepsilon, F(x+\varepsilon), F(-x))$$

$$(x \in (-\varepsilon, -\tfrac{1}{2}\delta] \cap I).$$

This is possible since

$$|\tfrac{1}{2}g(-\tfrac{1}{2}\delta)| \le \tfrac{1}{2}F(\tfrac{1}{2}\delta) < F(\tfrac{1}{2}\delta) < F(-\tfrac{1}{2}\delta+\varepsilon) .$$

For x non-negative we define

$$g_1(x) = \begin{cases} s_{2k+1}(x) & \text{if} \quad c_{2k+1} < x \le \tfrac{1}{2}b_{2k+1} , \\ g(x) & \text{if} \quad \tfrac{1}{2}b_{2k+1} < x \le c_{2k} , \\ g(x)-s_{2k}(x) & \text{if} \quad c_{2k} < x \le \tfrac{1}{2}b_{2k} \end{cases}$$

$$(k = 0, 1, \ldots)$$

and $g_1(x) = 0$ otherwise;

$$g_2(x) = \begin{cases} s_{2k}(x) & \text{if} \quad c_{2k} < x \le \tfrac{1}{2}b_{2k} , \\ g(x) & \text{if} \quad \tfrac{1}{2}b_{2k} < x \le c_{2k-1} , \\ g(x)-s_{2k-1}(x) & \text{if} \quad c_{2k-1} < x \le \tfrac{1}{2}b_{2k-1} \end{cases}$$

$$(k=1,2,\ldots)$$

and $g_2(x) = 0$ otherwise. Finally we put

$$K_1 = (-\infty,-\varepsilon] \cup \{0\} \cup \bigcup_{k=1}^{\infty} [b_{2k}, c_{2k-1}] \cup [b_o,\infty)$$

and

$$K_2 = (-\infty,-\varepsilon] \cup \{0\} \cup \bigcup_{k=1}^{\infty} [b_{2k+1}, c_{2k}] \cup [b_o,\infty) .$$

Let $k \ge 0$ be fixed. g_1 is obviously differentiable at every point of $(c_{2k+1}, b_{2k}) - \{\tfrac{1}{2}b_{2k+1}, c_{2k}, \tfrac{1}{2}b_{2k}\}$. However

(14) implies that g_1 is differentiable also at the points $\frac{1}{2}b_{2k+1}$ and $\frac{1}{2}b_{2k}$. By (4) and (13) it follows that

$$\lim_{x \to c_{2k}+o} \frac{s_{2k}(x)}{x-c_{2k}} = 0$$

and hence g_1 is differentiable at c_{2k}, too. Thus we have proved that g_1 is differentiable at the points of $I-K_1$.

If $x \in (c_{2k+1}, \frac{1}{2}b_{2k})$ then $\mathrm{dist}(x,K_1) = x-c_{2k+1}$. Hence, for $c_{2k+1} < x \leq c_{2k}$ we have

$$|g_1(x)| \leq 2\cdot F(x-c_{2k+1}) = 2\cdot F(\mathrm{dist}(x,K_1))$$

by (13), (10) and (11). Similarly, if $c_{2k} < x \leq \frac{1}{2}b_{2k}$ then

$$|g_1(x)| \leq |g(x)| + |s_{2k}(x)| \leq F(x) + 2\cdot F(x-c_{2k}) \leq$$
$$\leq 2\cdot F(x-c_{2k+1}) + 2\cdot F(x-c_{2k+1}) = 4\cdot F(\mathrm{dist}(x,K_1))$$

This proves that $|g_1(x)| \leq 4\cdot F(\mathrm{dist}(x,K_1))$ holds in I. This estimation, together with (4) implies $g_1'(x) = 0$ for every $x \in I \cap K_1$ and hence $g_1 \in \mathcal{D}$. Using (10) we obtain $|g_1| < \varepsilon$, too. These assertions can be similarly proved for g_2 and K_2. Since $g_1(x)+g_2(x) = g(x)$ holds in $[-\frac{1}{2}\delta, b_1]$ and (9) follows immediately from (12) the proof of the Lemma is complete. □

Proof of Theorem 2. Let H denote the set of those points $a \in I$ for which there exists $f_a \in \mathcal{D}$ such that

$$(df_a)(a) \neq h(a) \cdot f_a'(a) .$$

We have to prove that H is scattered. Suppose this is not true and let $A \subset H$ be non-empty and dense in itself. We can choose A to be countable; otherwise we take a countable and dense subset of A. Making use of this assumption we shall construct a function $\varphi \in \mathcal{D}$ such that $d\varphi$ is not Baire 1.

For $a \in A$ we put

$$g_a = f_a - f_a(a) - f_a'(a)(\sigma - a) ,$$

then $g_a(a) = g_a'(a) = 0$ and

$$(dg_a)(a) = (df_a)(a) - f_a'(a) \cdot h(a) \neq 0 .$$

Multiplying by a suitable constant we can suppose that

$$(15) \qquad (dg_a)(a) \geq 2 \qquad (a \in A) .$$

Let $A = \{a_n\}_{n=1}^{\infty}$ and define

$$F_n(x) = \sup_{\substack{|y| \leq x \\ a_n + y \in I}} |g_{a_n}(a_n + y)| \qquad (x \geq 0, \quad n=1,2,\ldots).$$

Then F_n is continuous and increasing on $[0,\infty)$. In addition it follows from $g_{a_n}(a_n) = g_{a_n}'(a_n) = 0$ that

$$\lim_{x\to+0} \frac{F_n(x)}{x} = 0 \qquad (n=1,2,\ldots).$$

Let $\delta_n > 0$ be chosen according to

$$F_n(x) < \frac{1}{2^n}, \qquad \frac{F_n(x)}{x} < \frac{1}{2^n} \qquad (0 < x \le \delta_n)$$

and put

(16) $$F(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \min(F_n(x), F_n(\delta_n)) \qquad (x \ge 0).$$

Then F is continuous and increasing on $[0,\infty)$;

(17) $$F(0) = F'_+(0) = 0$$

and

(18) $$|g_{a_n}(x)| \le F(|x-a_n|) \qquad (|x-a_n| < \delta_n)$$

for every $n=1,2,\ldots$.

Let $p(n)$ denote the greatest integer k such that $3^k | n$. Then $0 \le p(n) < n$ for every $n=1,2,\ldots$ and for each non-negative integer k there are infinitely many even as well as odd natural numbers n with $p(n) = k$.

Now we turn to the construction of the function φ . Let $t_0 \in A \cap \operatorname{int} I$ be arbitrary and put $\varphi_0 \equiv 0$ and $P_0 = \mathbb{R}$. Let $n > 0$ and suppose that the points $t_0, t_1, \ldots, t_{n-1} \in A \cap \operatorname{int} I$, the function $\varphi_{n-1} : I \to \mathbb{R}$ and the closed set P_{n-1} have been define in such a way that t_i is an accumulation point of $\operatorname{int} P_{n-1} \cap A$ for every $0 \le i \le n-1$.

Then we choose a point $t_n \in (\text{int } P_{n-1} \cap A \cap \text{int } I) - \{t_i\}_{i=0}^{n-1}$ such that

$$(19) \qquad |t_n - t_{p(n)}| < \frac{1}{n} .$$

Next choose $\varepsilon_n > 0$ so that

$$(20) \qquad [t_n - \varepsilon_n, t_n + \varepsilon_n] \subset P_{n-1} - \{t_i\}_{i=0}^{n-1}$$

and

$$(21) \qquad \varepsilon_n \le |x - t_i|^3 \quad \text{for every} \quad x \in [t_n - \varepsilon_n, t_n + \varepsilon_n] \quad \text{and} \quad i = 0, 1, \dots, n-1 .$$

Now we apply the Lemma with F defined in (16), $g = g_{t_n}$, $a = t_n$ and with $\varepsilon = \varepsilon_n$. Then the conditions of the Lemma are satisfied by (17) and (18). (Observe that $a = t_n$ is an accumulation point of A since A is dense in itself.) Thus, by the Lemma, we are given the functions g_1, g_2 and closed sets K_1, K_2 satisfying (5)-(9). Now (5) asserts that $g_1 + g_2 - g_{t_n}$ vanishes in a neighbourhood of t_n and hence, by Theorem 1,

$$(d(g_1 + g_2 - g_{t_n}))(t_n) = 0 .$$

From this, applying (15) we obtain

$$(dg_1)(t_n) + (dg_2)(t_n) = (dg_{t_n})(t_n) \ge 2$$

and thus we have either

$$(dg_1)(t_n) \ge 1 \quad \text{or} \quad (dg_2)(t_n) \ge 1 .$$

Suppose e.g. $(dg_1)(t_n) \ge 1$ and put

$$\varphi_n(x) = \begin{cases} \varphi_{n-1}(x) & \text{if} \quad x \in I - [t_n-\varepsilon_n, t_n+\varepsilon_n] , \\ \frac{(-1)^n+1}{2} \cdot g_1(x) & \text{if} \quad x \in I \cap [t_n-\varepsilon_n, t_n+\varepsilon_n] \end{cases}$$

and $P_n = P_{n-1} \cap K_1$. It follows from (20), (8) and (9) that t_i is an accumulation point of $\operatorname{int} P_n \cap A$ for each $0 \le i \le n$. In this way we have defined by induction the sequence $\{t_n\}_{n=0}^{\infty} \subset A$, the functions φ_n $(n=0,1,\ldots)$ and the closed sets $P_0 \supset P_1 \supset \ldots$.

The definition of φ_n and (20) imply that $\varphi_n(x) = \varphi_{n-1}(x)$ holds for every $x \in I-P_{n-1}$ $(n=1,2,\ldots)$. Now we define φ by

$$\varphi(x) = \begin{cases} \varphi_n(x) & \text{if} \quad x \in I-P_n \qquad (n=0,1,\ldots) , \\ 0 & \text{if} \quad x \in I \cap \bigcap_{n=1}^{\infty} P_n . \end{cases}$$

We contend that $\varphi \in \mathfrak{D}$ and $d\varphi$ is not Baire 1. It easily follows by induction from (7), (8) and $P_{n-1} \supset P_n$ that

(22) $$|\varphi_n(x)| \le 4 \cdot F(\operatorname{dist}(x,P_n)) \qquad (x \in I, \ n=0,1,\ldots).$$

Hence we have $\varphi_n(x) = \varphi_n'(x) = 0$ for every $x \in P_n$ which implies (also by induction) that $\varphi_n \in \mathfrak{D}$ for every $n=0,1,\ldots$ Thus φ is differentiable at every point of $I-P$, where

$P = \bigcap_{n=1}^{\infty} P_n$. Since (22) implies $|\varphi(x)| \le 4 \cdot F(\mathrm{dist}(x,P))$ $(x \in I)$ hence $\varphi(x) = \varphi'(x) = 0$ for $x \in I \cap P$ and thus we have $\varphi \in \mathfrak{D}$.

Next we show that

$$(23) \qquad \begin{aligned} (d\varphi)(t_k) &= 0 \quad \text{if} \quad k=1,3,5,\ldots , \\ (d\varphi)(t_k) &\ge 1 \quad \text{if} \quad k=0,2,4,\ldots . \end{aligned}$$

First we prove

$$(24) \qquad \lim_{x \to t_k} \frac{\varphi(x)-\varphi_k(x)}{(x-t_k)^2} = 0 \qquad (k=0,1,\ldots).$$

Let $k \ge 0$ be fixed and let $x \in I$ be such that $\varphi(x)-\varphi_k(x) \ne 0$. Then $x \in P_k$, since otherwise $\varphi(x) = \varphi_k(x)$ would hold, and thus $\varphi_k(x) = 0$ and $\varphi(x) \ne 0$. This implies that there is $n > k$ such that $|x-t_n| \le \varepsilon_n$ and $\varphi(x) = \varphi_n(x)$. Now it follows from (6), (21) and from the definition of φ_n that

$$|\varphi(x)-\varphi_k(x)| = |\varphi(x)| = |\varphi_n(x)| \le |x-t_k|^3 ,$$

which proves (24).

By Theorem 1, (24) implies that $(d(\varphi-\varphi_k))(t_k) = 0$, from which we obtain $(d\varphi)(t_k) = (d\varphi_k)(t_k)$. If k is odd, then φ_k vanishes in $[t_k-\varepsilon_k, t_k+\varepsilon_k]$ and hence, using Theorem 1 again, $(d\varphi_k)(t_k) = 0$. If k is even then, by the definition of φ_k , there exists a function $g \in \mathfrak{D}$ such that $(dg)(t_k) \ge 1$ and φ_k-g vanishes in $[t_k-\varepsilon_k, t_k+\varepsilon_k]$.

Therefore $(d\varphi_k)(t_k) = (dg)(t_k) \geq 1$ and thus (23) is proved.

Let Z denote the closure of $\{t_k\}_{k=o}^{\infty}$. It follows from (19) that for every $k \geq 0$ and $\delta > 0$ there are infinitely many even as well as odd natural numbers n such that $t_n \in (t_k-\delta, t_k+\delta)$. Hence both $(d\varphi)(x) = 0$ and $(d\varphi)(x) \geq 1$ holds in a dense subset of Z and thus $d\varphi|_Z$ cannot have any point of continuity. Therefore, φ is not Baire 1, contradicting our assumption. This contradiction completes the proof of Theorem 2. □

We remark that for every scattered set $H \subset I$ there exists a derivation d on $\mathcal{D}$ such that df is Baire 1 for every $f \in \mathcal{D}$ and for each $a \in H$ there is $f_a \in \mathcal{D}$ with

$$(df)(a) \neq h(a) \cdot f'(a) .$$

In fact, let h be a fixed Baire 1 function on I. For every $a \in I$ we can choose a linear functional $\lambda_a : \mathcal{D} \to \mathbb{R}$ such that $\lambda_a \not\equiv 0$ and $\mathcal{P}_2(a) \subset \operatorname{Ker} \lambda_a$ if $a \in H$, and $\lambda_a \equiv 0$ if $a \in I-H$. Then we define $d: \mathcal{D} \to \mathbb{R}^I$ by

$$(df)(a) = h(a) \cdot f'(a) + \lambda_a f \qquad (f \in \mathcal{D}, \quad a \in I).$$

Then d is a derivation by the Corollary of Theorem 1. Let $f \in \mathcal{D}$ be fixed and denote $g = df - h \cdot f'$. The set $\{x \in I, g(x) \neq 0\} \subset H$ is scattered and this easily implies that g is Baire 1. Since $h \cdot f'$ is also Baire 1, the same is true for

$$df = h \cdot f' + g$$

which proves our assertion.

References

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