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CLASSICAL THEORY OF TOTALLY IMPERFECT SPACES

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Section I. Introduction.

Section 40 of Kuratowski's Topology Vol. 1 [Ku66] is a study of certain singular separable metric spaces. One group of spaces, L, ν, concentrated (or P), C", and C spaces are "sparse" in the measure theoretic sense, in that every such subset of the reals has Lebesgue measure 0, and are related to the so-called Lusin set (an uncountable subset of the reals R, every nowhere dense subset of which is countable) [Lu14]. The other group of spaces studied there, σ, λ, λ', and always first category (AFC) spaces are "sparse" in the Baire category sense in that every such space is necessarily first category, and are related to the so-called Sierpinski set (an uncountable subset of R, every Lebesgue measurable subset of which is countable) [Si24]. It is shown that

$$(1.1)\quad \begin{array}{l} \text{Countable} \to L \to \nu \to P \to C'' \to C \to \text{TI} \\ \text{Countable} \to \lambda' \to \lambda \to \text{AFC} \to \text{TI} \\ \text{Countable} \to \sigma \to \lambda \end{array} \quad \text{for separable}$$

metric spaces, where "TI" means totally imperfect (i.e. contains no subset which is homeomorphic to a Cantor set). It is shown (using the Continuum Hypothesis, CH, in many cases) that most of the implications are irreversible. For each property, certain questions are considered. (1) Is the property hereditary? (2) Under what kinds of mappings is the property preserved? (3) Is the property finitely or countably additive? (4) Is the property preserved under taking products? (5) What are the possible dimensions of such spaces? (6) What can the "Baire order" of such spaces be (i.e. in which Baire class do all the Borel functions fall)? This last problem is of course related to the nature of the Borel subsets of the space as well as the sets with the Baire properties. A subset A of a space X has property B_w(rel X) if there exists an open set Q such that $A-Q$ and $Q-A$ are of first category. A has property B_r(rel X) if for every subset B of X, $B \cap A$ has property B_w(rel B) (see [Ku66, Sec. 11] for details). We say that a function f has property B_w (B_r) if for every closed subset P of the range of f, $f^{-1}(P)$ has property B_w (B_r) (rel domain of f).

The purpose of this paper is to give an expanded version of the exposition given in Section 40 of [Ku66]. There are many singularity properties which are related to those mentioned above which have been studied since 1966, and there are others which were studied earlier but were not considered in [Ku66]. There are also Theorems related to the properties of (1.1) which were not discussed in [Ku66]. We hope to present in an organized fashion a summary of a large percentage of the known relationships between such singularity properties and the answers to questions (1) - (6) and related questions which are known concerning these properties. No proofs will be given for the known results, although the original proofs may be discussed briefly in some cases. As the exposition proceeds, certain problems will present themselves. Solutions will be provided where the authors are able to do so. Other problems will be left open. The organization will be as follows. Sections 2 through 8 will consist of a detailed discussion of the properties indicated in (1.1) as well as closely related properties which were not considered in [Ku66].

Section 9 consists of a discussion of the interrelationships that are possible between the "Lusin branch" of (1.1) and the "Sierpinski branch". In Section 10 we present some applications of the theory of these singular sets to (1) the study of functions of a real variable and (2) the study of properties of the space $P(X)$ of probability measures on X. This theory has found applications in almost all branches of mathematics, but these are the areas where the authors have made a contribution.

By the "classical theory" of singular spaces, we mean the study of these singularity properties as they pertain to <u>separable metric spaces</u>, and all spaces are assumed to have this property in this paper. We will assume the ordinary axioms of set theory (ZFC) plus CH when necessary. There has been recent interest in the study of some of these properties in more general topological settings and in the construction of pertinent examples under set theoretic assumptions other than CH. But this paper will be primarily concerned with the classical theory, which relied mainly on CH. We will, however, mention results we know of which follow from consistent axioms other than CH and which <u>contradict</u> results which follow from CH.

Section II. Lusin sets, concentrated sets, and related properties.

A subset M of a space X has <u>property L (rel X)</u> if every nowhere dense subset of X intersects M in an at most countable set. By well-ordering the Cantor subsets of $[0,1]$ into a transfinite sequence of length $\aleph_1$ (assuming CH) and for each countable ordinal α, choosing a point $m_\alpha \in [0,1]$ which is not in any of the first α such Cantor sets, one can construct an uncountable set $\{m_\alpha\}$ with property L(rel R).

Lusin [Lu14] established (with CH) the existence of uncountable sets with property L relative to the reals R, and such sets have historically been referred to as "Lusin sets". Actually, P. Mahlo described such a set in [Ma13] (see Aufgabe 5), but this fact was overlooked historically, possibly because the result was not mentioned in the review of [Ma13] which appeared in Fort. Math. 44 (1913) p. 92. We thank Prof. John Morgan for informing us of this early paper of Mahlo. A space has property ν if it has property L relative to itself. Of course, a set M situated in a space X can have property ν and not have property L(rel X) [Ku66] [KuSi36], for example, a Lusin set can be constructed as described above inside a fixed Cantor set.

The set $A \subseteq X$ is concentrated about a set $B \subseteq X$ if every open set containing B contains all but at most countably many elements of A. A subspace Y of a space X is said to be con(rel X) if it is concentrated about a countable subset of X, and a space is said to have property P if it is concentrated about a countable subset of itself. The notion of "concentration" was defined by Besicovitch [Be34].

Useful characterizations of property ν are the following: (1) X is homeomorphic to a set which has property L(rel R) [KuSi36], and (2) X is concentrated about every countable subset of itself ([Ku66] and [Sz38]). Both properties L(rel X) and ν are obviously hereditary, and property L(rel X) is preserved in countable unions. Property ν is not preserved under taking finite unions [Co81a]. However, if $M_1, M_2, \ldots$ are ν spaces and each M_i is dense in $X = M_1 \cup M_2 \cup \ldots$, then X is a ν space [Co81a]. Since a set M which is L(rel R) can be transformed homeomorphically into a subset of a Cantor set, property L(rel X) would not be preserved under homeomorphisms. But if M is L(rel X) and f is a homeomorphism from X onto Y, then f(M) is L(rel Y). By constrast, it is known (under CH) [Lu33] that for X = the irrationals, there is a 1-1 continuous $f: X \to X$ which transforms every set with property L(rel X) onto a set with property AFC, so this image cannot even have property ν.

We now define two properties, L_1 and $C(\nu)$, which were not considered in [Ku66]. Countable unions of ν spaces are not necessarily ν spaces and were called L_1 spaces in [Br74], where they played an essential role in the characterization of spaces in which certain variants of Blumberg's theorem hold (to be discussed further in Section X). It is shown in [Ku66] that every continuous image of a ν space has property C", it is shown in [Da69c] that Borel images of ν spaces have property β, and it was pointed out in [Ro38] that the B_W image of a ν space has property C'. Actually a space is the B_W image of a ν space if and only if it is the continuous image of a ν space and this latter property, called property $C(\nu)$ in [BrCo82], fits properly between L_1 and property P, so all of the properties in

(2.1) countable $\to$ L(rel X) $\to \nu \to L_1 \to C(\nu) \to$ P $\to$ con(rel X)

are different (assuming CH). It is obvious that all of the above properties except P are hereditary, and all except L(rel X) and con (rel X) are preserved under homeomorphisms (indeed $C(\nu)$ and P are preserved under continuous mappings). Likewise, properties L_1, $C(\nu)$, P and con(rel X) are obviously preserved under taking countable unions. The question of whether con(rel X) is preserved under homeomorphisms was rather more difficult, being settled (under CH) in the negative by Rothberger [Ro41]. It was also shown (under CH) that every number set of cardinality c is the continuous image of a con(rel R) set, but that property con(rel R) is preserved by continuous functions from R into R.

As far as products are concerned, it was shown (using CH) by Sierpinski in [Si35] that there is a set M with property L(rel R) such that the "difference set" $D = \{x-y | x \in M, y \in M\} = R$. It follows that the projection of M^2 on the line "$y = -x$" is the entire line, so that M^2 cannot have property C, which is weaker than con(rel X). Thus none of the properties of (2.1) is preserved under taking products.

Sierpinski's difference set D of the previous paragraph is a "vector sum" of two Lusin sets. There has been recent interest in singularity properties defined in terms of such vector sums. If A and B are subsets of the reals R, then $A + B = \{a + b | a \in A, b \in B\}$. Talagrand [Ta75] studied sets $A \subseteq R$ which satisfy what we will call property (T): for every compact set K with $\lambda(K) = 0$, we have $\lambda(A+K) = 0$, where λ is Lebesgue measure (Talagrand actually dealt with subsets of a locally compact group and Haar measure). He showed that no uncountable analytic set satisfies (T). It follows that (T) $\rightarrow$ TI. But he showed that CH implied the existence of a set B with cardinality c which satisfies (T) and is such that for every dense G_δ subset X of R, $X + B = R$. Then, Friedman and Talagrand [FrTa80] proved a theorem using Martin's axiom (MA), which under the stronger axiom CH yields the existence of a subset A of R such that (1) $|A| = c$, (2) for every subset X of R for which $\lambda(X) = 0$, $\lambda(A+X) = 0$, (3) for every first category subset X of R, $A+X$ is first category, (4) A is concentrated about the dydadic rationals, and (5) A is AFC (see Section VII). We will refer to property (2) above as <u>Property (T+)</u>. The fact that (T) $\rightarrow$ TI was strengthened in [EKM81], where it was shown that if P is a perfect subset of R, there is actually a perfect subset M of E with $\lambda(M) = 0$ and $P+M = R$ (in which case $M \times P$ has infinite linear measure). It was also shown that con(rel R) $\rightarrow$ (T), and examples related to the above mentioned examples of Sierpinski, Friedman, and Talagrand are given. In particular, it was shown that condition (T+) for sets $A \subseteq R$ is implied by the following: <u>(T*)</u> for every subset X of R with $\lambda(X) = 0$, $X \times A$ has linear measure zero. The question of whether (T+) = (T*) is raised. They show that CH implies the existence of an uncountable subset of R which satisfies (T*) and is concentrated about the rationals. Note that while (T) is intermediate to con(rel R) and TI, (T+) is not because of the Lusin set of Sierpinski which has difference set R. ((T) and analytic) is impossible for subsets of R, but they show that Godel's "V = L" axiom implies the existence of an uncountable coanalytic set which satisfies (T+) and is concentrated about the rationals. For other examples of singular coanalytic or projective sets constructed under axioms other than CH, see [Ku48], [Os75].

It was shown in [Sz37a] [Ku66] that any set with property C must be 0-dimensional, so all of the properties in (2.1) imply 0-dimensionality.

Telegarsky [Te72] has defined a sequence of properties which are generalizations of (and weaker than) property P. If n is a positive integer, we say that X is <u>n-chain-concentrated</u> (<u>n-CC</u>) if there is a sequence $X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n$ such that X_n is countable and X_k is concentrated about X_{k+1} for each $0 \leq k \leq n-1$ (so that "1-CC" is P). He showed that $P \rightarrow$ n-CC $\rightarrow$ n+1-CC $\rightarrow$ C" (C" is stronger than C and is discussed in the next section), and that property n-CC is preserved under taking countable unions. He also showed that if X has <u>property P^m</u> (i.e. for some countable set $A \subseteq X$, X^m is concentrated about $X^m-(X-A)^m$), for each $1 \leq m \leq n$, then X^n is n-CC, with the countable set being A^n. This is related to an example of Michael [Mi71] and to later work of Cox, [Co80] and [Co81a] which will be discussed later in this section.

Gardner [Ga79] defined a sequence of properties, <u>B_1, B_2, ...</u> B_α, ..., which are generalizations of (and weaker than) con(rel X). Let Y be a subspace of X. B_1 is just con(rel X). If α is a non-limit ordinal or an ordinal which is not cofinal with ω_0, we say that Y has property B_α if there is a set $A \subseteq X$ with property B_β for some $\beta < \alpha$ such that if G is an open set containing A, then $Y-G$ has property B_γ for some $\gamma < \alpha$. If α is a limit ordinal which is cofinal with ω_0, we say Y has property B_α if Y is a countable union of previously defined sets. All of these properties are (rel X). He showed that con $\to B_\alpha \to B_{\alpha+1} \to C$ for $\alpha \leq \omega_2$, and that if $\omega_2 = \alpha < \beta$, then $B_\alpha = B_\beta$. He also showed that $B_n \not\leftrightarrow B_{n+1}$ if $n < \omega_0$, and that the properties B_α are hereditary and preserved under taking countable unions. The properties are related to Telgarsky's properties as follows: (2^n-1)-CC $\to B_n$.

The entire study of singular spaces has been closely related to the so-called "Baire order problem". The Baire order of a space X is defined as follows. Let $G_0, G_1, \ldots, G_\alpha, \ldots$ be the usual transfinite sequence with union the Borel sets, where G_0 contains the open sets, G_1 contains the G_δ sets, G_2 contains the $G_{\delta\sigma}$ sets, etc. The Baire order of X is the first ordinal α such that $G_\alpha = G_{\alpha+1}$ (Note: in some cases the listing of the Borel classes starts with G_1 containing the open sets, so that the finite Baire order numbers are shifted by 1). The reals have Baire order ω_1. [Ku66, Sec. 3], and countable spaces have Baire order ≤ 1 [Si30]. The "Baire order problem of Mazurkiewicz" [Po30] was to determine whether it is true for each countable ordinal α, there is a space with Baire order α. The Sierpinski set was shown by Szpilrajn [Sz30b] to have Baire order 1, and Poprugenko [Po30] [Ku66, p. 526] showed that an uncountable ν space has Baire order 2. Poprugenko stated in 1930 that the general problem "remains unsolved and appears to be very difficult even for $\alpha = 3$". Indeed, the problem remained unsolved (even for $\alpha = 3$) until very recently, when Miller and Kunen [MK79] verified (under CH) the existence of a space with Baire order α for every countable ordinal α (the "shifted" counting system is used herein).

In proving that ν spaces have Baire order ≤ 2, it is actually possible to show that B_w sets must be the union of a G_δ and a countable set [Ku66, p. 526]. It was shown in [Br77b] that in L_1 spaces, even B_r sets are not necessarily the union of a G_δ and an F_σ, but that B_r sets are necessarily $G_{\delta\sigma}$ sets, so that L_1 spaces also have Baire order ≤ 2. But an example (using CH) was given of a P space which has Baire order ≥ 3. It was not determined what the exact Baire order of that space was.

Brown and Gardner [BrGa79] have defined a sequence of properties which are generalizations of (and weaker than) properties ν and L_1. The motive behind the definitions was the hope of gaining further insight into the Baire order problem. The properties are similar to those of [Te72] and [Ga79] in that they lie intermediate to ν and C. Setting $\nu_1 = \nu$, define ν_2, L_2, ν_3, L_3, ... as follows. A ν_n space is one for which every nowhere dense subset is L_{n-1}, and an L_n space is the countable union of ν_n spaces. A ν_ω space is one for which every nowhere dense subset can be written as the countable union of sets, each of which is L_n for some n. It turns out that extension of the definitions to L_ω, $\nu_{\omega+1}$, ... etc. yields nothing new. It is shown that

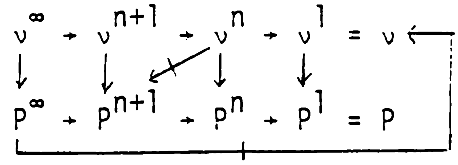
$$(2.2) \quad \nu = \nu_1 \rightarrow L_1 \begin{array}{l} \nearrow P = 1\text{-CC} \rightarrow i\text{-CC} \searrow \\ \searrow \nu_i \rightarrow L_i \rightarrow \nu_{i+1} \rightarrow \nu_\omega \nearrow \end{array} C'' \quad \text{for } 2 \leq i < \omega,$$

and that CH implies $i\text{-CC} \not\rightarrow \nu_{i+1}$ for $1 \leq i < \omega$, and that all the ν_i and L_i properties are different. The (CH) example of

[Br77b] shows that $P \not\to L_i$ for $1 \leq i < \omega$, and the question whether $P \to \nu_\omega$ is left open. The ν_α and L_α properties are obviously hereditary and preserved under homeomorphisms. Property ν_n is not preserved under taking countable unions, but properties L_n and ν_ω are. The L_n spaces all have Baire order ≤ 2 and ν_ω spaces have Baire order ≤ 3 (hopefully some ν_ω space might have Baire order = 3).

It should be noted that the Baire order of a space X can be defined equivalently as follows. Let B_0, B_1, ... be the transfinite sequence such that $B_0 = C(X)$, the collection of all continuous real valued functions with domain X, and for each α, B_α contains the functions which are pointwise limits of sequences of functions which come from previously defined classes. The Baire order of X (also called the Baire order of B_0) is again the first α for which $B_\alpha = B_{\alpha+1}$. There has been a great deal of interest in various "Baire order problems" different than the one referred to here, where B_0 is taken to be some class of functions other than C(X) for some space X, and the Baire order of B_0 is determined. The reader is referred to the work of Mauldin [Mn74], [Mn75] for an extensive study of various Baire order problems.

Cox [Co80][Co81a] has defined a sequence of properties which are generalizations of (and stronger than) properties L, ν, and P. For example, a space S is P^n if there exists a countable dense subset B of S such that S^n is concentrated about $S^n - (S-B)^n$. Hence, $P^1 = P$. Likewise, ν^n is defined, requiring concentration about every dense grid. A space is P^∞ or ν^∞ if it is P^n or ν^n for every positive integer n. It was shown in those two papers that

(2.3) 

<u>$H\nu^n$, HP^n, $H\nu^\infty$, and HP^∞</u> denote the properties of being <u>hereditarily</u> ν^n, P^n, ν^∞, and P^∞, respectively. Property ν^n $(n \geq 2)$ is not hereditary. In fact, a space can go from ν^∞ all the way to ν^1 by the removal of a countable set. P^n and HP^n spaces are preserved by continuous transformations, but ν^n spaces are not for essentially the same reason that ν^1 spaces are not. As for unions, $H\nu^n$, P^n, and HP^n spaces are preserved only in the weakest of senses. That is, when they are unioned with countable sets (a density requirement is also needed for $H\nu^n$ spaces), the result enjoys the same property. Even this weak preservation falls apart for ν^n spaces. An extension of Sierpinski's [Si35] example is given in [Co81a] where it is demonstrated that for each n, there is a $H\nu^n$ space for which the vector sum $\bigoplus_{i=1}^{n+1} S = R$. This implies that S^{n+1} does not have property C, so products here are rather sensitive. The dimension of ν^n and P^n spaces are, of course, zero, and the Baire order of ν^n spaces cannot exceed 2. It is believed, but not substantiated, that Brown's example [Br77b] of a P space with Baire order ≥ 3 can actually be made into a HP^∞ space.

Section III: Properties C" and C and related properties.

A space X with metric δ has property C(rel δ) if for every sequence $\{\lambda_n\}$ of positive numbers, there exists a sequence $\{x_n\}$ of elements of X such that $X \subseteq N_\delta(x_1,\lambda_1) \cup N_\delta(x_2,\lambda_2) \cup N_\delta(x_3,\lambda_3) \cup \ldots$, where $N_\delta(x,\lambda) = \{y \in X \mid \delta(x,y) < \lambda\}$. This property was defined by Borel [Bo19], and he called sets with this property "the sets whose asymptotic measure is lower than any series given in advance". This property is also called the property of "strong measure zero" [La76]. Besicovitch [Be34] showed that for subsets of the reals, con(rel X) $\rightarrow$ C $\rightarrow$ β (also see [Sz34], where Poprugenko is credited with showing C $\rightarrow$ β in general). Besicovitch also showed that property C for subsets of R is equivalent to the requirement that X have measure zero with respect to every "Hausdorff" measure m_ϕ (also see [Da69a] and [Da73] in this regard).

Property C" is a topological version of C defined by Rothberger [Ro38]. A space X has property C" if it is true that for every system $\{G(x,n) \mid x \in X, n=1,2,\ldots\}$ of open sets such that $x \in G(x,n)$ for each x and n, there necessarily exists a "diagonal sequence" $\{x_n\}$ of elements of X such that $X \subseteq G(x_1,1) \cup G(x_2,2) \cup \ldots$. Rothberger also defined property C', which is similar to C", except the "diagonal sequence" is required to exist only for those systems such that for fixed n, there are only finitely many different $G(x,n)$. He showed in [Ro38] that within the realm of σ-totally bounded spaces (which includes subspaces of the reals) (1) $\nu \rightarrow$ C" $\rightarrow$ C' $\rightarrow$ C and (2) properties C' and C" are invariant under continuous transformations and therefore countably additive (that $P \to C''$ follows from [Te72]). He left open at that time the question of whether $C'' = C' = C$ and raised the question of whether property C' is preserved by taking images under functions with the Baire property. We do not know whether the results concerning property C' carry over from σ-totally bounded spaces to separable spaces, but the results concerning property C'' do (see [Ku66]). In [Ro41], it is shown (under CH) that property P is not hereditary, $[0,1]$ is the continuous image of a con(rel R) set, properties C' and C'' are not hereditary, and property C is not preserved under homeomorphisms. Property C is preserved by transformation under uniformly continuous functions [Gr81], and for subsets of R, it is preserved by continuous transformations from R into R [Sz30a]. Rothberger [Ro41] showed (under CH) that every set of reals of power c is the continuous image of a con(rel R) set, but property C' is preserved under such transformations, it follows (under CH) that con(rel X) $\not\to C'$. Besicovitch showed (under CH) in [Be42] that $C \not\to$ con(rel X), and R. Gardner has shown (in an informal communication) that that example of Besicovitch's actually satisfies C'', so we have

$$(3.1) \qquad \begin{array}{ccccc} & \nearrow & \text{con(rel X)} & \searrow & \\ P & & \not\uparrow \quad \not\downarrow & & C, \\ & \searrow & C'' \longrightarrow C' & \nearrow & \end{array} \quad \text{in } \sigma\text{-totally bounded spaces.}$$

When we include the previously defined properties, we have the following for separable spaces

$$(3.2) \qquad \begin{array}{cccccccccccc} & \nearrow & L^\infty & \to & L^n & \to & L^1 & = & L & & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & & \\ \text{countable} & \to & \nu^\infty & \to & \nu^n & \to & \nu^1 & = & \nu & = & \nu_1 & \to L_1 \end{array}$$

and

$$\text{(3.3)}\qquad \begin{array}{ccccccccccccc} & & & & \mathrm{con}(\mathrm{rel}\ X) = B_1 & \rightarrow & B_i & & & & & \searrow & \\ & & & & \uparrow & & & & \uparrow & & & & \\ L_1 & \rightarrow & C(\nu) & \rightarrow & P = 1\text{-}CC & \rightarrow & i\text{-}CC & \rightarrow & (2^i-1)\text{-}CC & \rightarrow & C'' & \rightarrow & C \\ & \searrow & & & & & \uparrow & & & \nearrow & & & \\ & & \nu_i & & \rightarrow & & L_i & & \rightarrow & \nu_\omega & & & \end{array}$$

Property C" appears to be fundamentally different from property C in the sense that while the collection of all sequences $\lambda_1, \lambda_2, \ldots$ of positive numbers has cardinality c, the collection of systems $\{G(x,n)\}$ such as are used in the definition of C" has cardinality 2^c if X has cardinality c. However, the definition of property C" as stated here is equivalent to the requirement that for every "double sequence" $\{U_n^m | m,n=1,2,\ldots\}$ of open sets such that $X \subseteq U_n^1 \cup U_n^2 \cup \ldots$ for each n, there necessarily exists a sequence m(1), m(2), ... such that $X \subseteq U_1^{m(1)} \cup U_2^{m(2)} \cup \ldots$ [Ro41, p. 114], and the collection of all such "double sequences" has cardinality c. The similarity between C" and C is made even clearer by the following characterization of C" which is given in [BrCo81]: A space X (with metric δ) has property C" if and only if it is true that for every sequence $f_1, f_2, \ldots$ of positive valued continuous functions with domain X, there exists a sequence $x_1, x_2, \ldots$ of elements of X such that $X \subseteq N_\delta(x_1, f(x_1)) \cup N_\delta(x_2, f(x_2)) \cup \ldots$.

Properties in the Lusin branch are measure theoretically singular because they all imply property β, sometimes called "universal measure zero", and the properties in the Sierpinski branch are singular in the Baire category sense because they all imply property AFC. Lusin [Lu33] raised the question of just how measure theoretically "massive" it is possible for the AFC sets to be. The comparable question of just how categorically "massive" it is possible for the sets of Lusin type can be was considered in [BrCo81]. It is clear that an uncountable space X with property ν must be 2nd category and must therefore contain a subspace which is <u>Baire complete (BC)</u> (i.e. every open set is 2nd category with respect to itself). L_1 spaces can be 1st category. On the other hand, it is impossible for a space Y which is con(rel X) to be <u>strongly Baire complete (SBC)</u> (i.e. every closed set in Y is 2nd category relative to itself). However it was shown in [BrCo81] that (assuming CH) there is a space with property C" which is SBC.

We have already stated the results that property C implies 0-dimensionality and that (under CH) none of the properties in (3.2) and (3.3) are preserved under taking products.

<u>Section IV: Universal null sets - property β</u>.

The study of the so-called "universal (or absolute) null" sets, sets of "universal (or absolute) measure zero", or sets with "property β" has been a long and on-going one. A good summary of some of the early work on the subject was given in the "Annex" to Vol. I of <u>Fund</u>. <u>Math</u>., which was written in 1937 by S. Braun and E. Spilrajn-Marczewski in collaboration with C. Kuratowski. The pertinent part of the "Annex" is the response to "Problem 5" of Vol. I of <u>Fund</u>. <u>Math</u>., in which Sierpinski had asked whether there exists an uncountable set with property (ii) described below. It is shown in the "Annex" that the following properties for subsets A of $I = [0,1]$ are all equivalent:

(i) for every increasing function $f: R \to R$, $\lambda(f(A)) = 0$,

(ii) for every homeomorphism f from A onto a subset of R, $\lambda(f(A)) = 0$,

(iii) for every bi-measurable(B) 1-1 $f:A \to R$, $\lambda(f(A)) = 0$,

(iv) for every 1-1 $f:A \to R$ for which f^{-1} is Borel, $\lambda(f(A)) = 0$,

(v) for every complete continuous measure μ on I, $\mu(A) = 0$.

By a complete continuous measure on a space X we mean the completion of a nonnegative Borel measure on X which assigns zero to single element sets. Lusin and Sierpinski had shown (without CH) [LuSi18] that there is a set Q of cardinality $\aleph_1$ which satisfies (i). The set Q was constructed by considering a non-Borel co-analytic set $C \subseteq I$. Then $C = E_1 \cup E_2 \cup \ldots \cup E_\alpha \cup \ldots$ $\alpha < \Omega$, where Ω is the first uncountable ordinal, the E_α are disjoint Borel sets, and if B is a Borel subset of C, then B is a subset of a countable union of the E_α's. Q contains just one point from each of the "constituants", E_α, of C. Then, it was shown in [Si25] that the same set Q satisfies (ii) and in [Si34e] that Q satisfies (iii). It was shown in [SiSz36] that the following two conditions are equivalent for subsets A of a separable metric space X:

(α) for every complete continuous measure μ on X, $\mu(A) = 0$,

(β) for every complete continuous measure μ on A, $\mu(A) = 0$.

It was shown in [SiSz36] (without CH) that there is a subset of I of cardinality $\aleph_1$ which satisfies (iv), (v) = (α), and (β). That example was not the same as the previous Q, but was based instead on the so-called "$\Omega - \Omega^*$ gap" of Hausdorff [Ha36], which implies the existence of a transfinite sequence

$B_1, B_2, \ldots, B_\alpha, \ldots \alpha < \Omega$ of disjoint Borel subsets of I such that if μ is any Borel measure on I, there is α such that $\mu(B_\alpha \cup B_{\alpha+1} \cup \ldots) = 0$. The example contains just one point from each B_α. These "real" examples based only on ZFC have cardinality $\aleph_1$. The question of whether there is a "real" example of a β set of cardinality c was considered in the "Annex". It was pointed out that CH examples with cardinality c were constructed in [Lf24] and [Sz34]. We now know that it will not be possible to exhibit such an example just under ZFC because Laver [La76] pointed out that it is consistent that it be true that every β space is of cardinality $\aleph_1$. Thus, we will see no "real" examples of β sets of cardinality c. On the other hand, since (MA + not CH) has been used to establish the existence of a β set of cardinality c (see the remarks preceding the theorem in [GrRy31]), we will see no "real" proof that every β set is of cardinality $\leq \aleph_1$.

It is natural at this point to wonder why no "real" examples of uncountable ν spaces or even C spaces have appeared in the literature. Kunen [Kn76] showed that for a fairly general class of topological spaces X, (MA + not CH) implies there will exist no uncountable sets with property L(rel X). Then, Laver [La76] showed that if ZFC is consistent, then so is (ZFC + every set with property C is countable). In view of this result, we see that no "real" examples of uncountable spaces with any of the properties in (3.2) - (3.3) will be found.

The study of β sets has been closely related to the so-called "problem of measure" of Banach, namely the question of whether or not there is a cardinal κ which is what is now called "real valued measurable", i.e. for which there exists a non-negative measure defined for <u>all</u> subsets of κ which assigns zero to singletons but is not identically zero. Banach and Kuratowski showed (under CH) [BaKu30] that c is not such a cardinal, and Ulam [Ul30] showed (without CH) that $\aleph_1$ is not one either (also see [Ox71, Sec. 5]). When it was shown (without CH) in [SiSz36] that there is a β set of cardinality $\aleph_1$, this of course also settled the Banach problem for $\aleph_1$. For recent results and further references concerning generalizations of this problem, see [Gr80].

That $C \to \beta$ is due to Besicovitch [Be34] for subsets of R, and to Poprugenko (see [Sz34]) for the separable metric case. There is a wealth of literature (assuming CH) showing that $C \nrightarrow \beta$. The first example that we know of, given in [MzSz37] was a β set in R^{n+1} of dimension n (sets with property C must have dimension 0 [Sz37a]). An eariler example would have been the square of the Lusin set of Sierpinski [Si35] with a square which is not C, because property β is preserved under taking products [Sz37b].

The (CH) proof in [MzSz37] of the existence of a β set in R^{n+1} of dimension n was based upon (1) the theorem of Hilgers (see [Ku66, p. 302]) which guarantees that for every n, every linear set of cardinality c is the 1-1 continuous image of a subset of R^{n+1} of dimension n, and (2) the fact that property β is preserved under 1-1 mappings f for which f^{-1} is a Borel function [SiSz36].

Other examples which show $C \nrightarrow \beta$ (recent ones based on ZFC) are also based upon theorems concerning preservation of property β under various kinds of mappings. It follows from (2) of the previous paragraph that the graph of an arbitrary function with domain a β set would have property β. Since projection from a product space onto one of the axes is continuous, it follows that an arbitrary space of the same cardinality as a β space is the 1-1 continuous image of a β space. From this and CH it follows that property β is not preserved under 1-1 continuous transformations. This idea was carried further in [Si38] where it was shown (under CH) that property β for subsets of the unit interval I is not even preserved under a continuous function h from I into I, and since property C is preserved under such transformations [Sz30a], this shows (under CH) that $C \nrightarrow \beta$. Later, Darst [Da70a] strengthened this result by making the function h also of bounded variation.

The CH assumptions in the results in the previous paragraph have recently been removed by E. Grzegorek. Grzegorek proved (in ZFC) [Gr80, Cor. 2] that there exist two subsets A and B of I such that $|A| = |B|$, A is β, and B has positive outer Lebesgue measure (and is therefore not β). Then the results of Sierpinski and Darst discussed in the previous paragraph were improved and established without CH in [Gr81]. In particular, it was shown in ZFC that (i) there exists a continuous $f: I \to I$ which transforms a β subset of I into a set which is not Lebesgue measurable, and (2) there exists a C^∞ function $f: I \to I$ which transforms a β set into a set which is not "universally measurable (rel I)" (defined below) and therefore not β. The latter result also improves a result of Darst which appeared in [Da71b]. In connection with (2), Grzegorek showed that functions of bounded variation from I into I would necessarily transform β sets into sets of Lebesgue measure zero.

The question as to whether or not "vector sums" of β subsets of R have property β was answered (under CH) by Sierpinski [Si35] (and others [Da65], [EKM81]) who described a Lusin set which has difference sets equal to R. Grzegorek has recently informed us (in personal communication) that he can now answer the question without CH, by showing that the set M which is the range of the function mentioned in (ii) of the previous paragraph, while not universally measurable, is the vector sum of two β sets. He did not see how to make M have positive Lebesgue measure, but pointed out that it will not be possible to make $M = R$ (in ZFC) because of the previously discussed results of Laver [La76].

In [GrRy81], Grzegorek and Ryll-Nardzewski study the spaces X for which (i) there exists a β subspace Y of X of the same cardinality as X. They show (among other things) that for spaces X which are not TI, (i) will hold if and only if (ii) there exists a permutation p of X (a 1-1 function from X onto X) such that $p = p^{-1}$ and the graph of p is β. They asked if there exists a Sierpinski set with property (ii). Cox described such an example (using CH) in [Co81b]. Since a Sierpinski set X which is the domain of that function p must be nonmeasurable, it cannot have property C. But it is a uniformly continuous image of the graph, so the graph is another space which has property β but not C.

Finally, we will discuss the relationship between the β sets and the "universally (or absolutely) (or perfectly) measurable" sets. The first extensive study of these sets was carried out by Szpilrajn-Marczewski in a paper [Sz37b] which was written in Polish and published in C.R. Soc. Sci. Lett. Varsovie in 1937, and has not been generally available to many mathematicians. We thank John C. Morgan for providing us with an English translation of this important paper.

Following Szpilrajn-Marczewski, we will say that a subset A of a space X has property M(rel X) if for every Borel measure μ on X, there exist Borel sets $B_1 \subseteq A \subseteq B_2$ with $\mu(B_1) = \mu(B_2)$, and we say that A is "universally (or absolutely) measurable" or that A has property M if A has property M(rel X) for every space X in which A can be embedded. [Sz37b] contains an extensive study of these properties as well as the "absolutely measurable" functions. Of primary interest to us are the following results. If X is a space, then the class of subsets of X with property M(rel X) forms a σ-algebra which is closed under operation (A), so that it contains the class of analytic subsets of X, and if f is a bi-measurable(B) function from X into Y, f transforms sets with property M(rel X) onto sets with property M(rel Y). It is shown that A has property M if and only if A has property M(rel X) for some complete space X in which A can be embedded. Property M is preserved under bimeasurable(B) mappings. If A has property M(rel X), then $A \times Y$ has property $M(\text{rel } X \times Y)$. It is shown that the following property for subsets A of a space X,

(o) for every space X in which A can be embedded and every continuous complete measure μ on X, $\mu(A) = 0$,

is equivalent to properties (α) and (β), given above, and also to each of the following properties,

(γ) every subset of A has property M, and

(δ) A is TI and has property M.

It is shown that the β subsets of a space X form a σ-ideal, and that the product of two sets is a β set if and only if both components are. [Sz37b] also includes an expository discussion of many of the results about property β known at that time, as well as comparisons with property AFC and property (s^0), discussed below. It is also shown that Hurewicz's property,

(H) X is uncountable and every 0-dimensional subset of X is countable,

implies property β (Hurewicz [Hu32] had shown that CH implied the existence of a subset of Hilbert space with property H, and Hausdorff had shown [Ha36] that $H \to$ AFC).

Recall that the Lebesgue measurable subsets of R are the sets whose symmetric difference with some Borel set is of Lebesgue measure zero. It was shown in [Sz55] (under CH) that the subsets of an uncountable complete space X with property M are not just the sets (M_0) whose symmetric difference with some Borel set has property β. In particular, it was shown that (under CH) there will exist a collection with cardinality 2^c of M sets, no two of which have symmetric difference a β set. This result was improved in [GrRy80], where it was shown (without CH) that there will actually exist an analytic set which does not have property M_0.

Spilrajn-Marczewski's result that universal measurability is an intrinsic invariant relative to complete spaces says that the question of whether or not a space X is universally measurable depends only on the topology for X. R.M. Shortt [Sh82a] has recently shown that it actually only depends only upon the Borel structure of X. In particular, he showed that if d_1 and d_2 are two metrics for X which induce the same Borel structure on X, then X has property M(rel the d_1-completion of X) if and only if X has property M(rel the d_2-completion of X). Then, in [Sh82a], [Sh82b], and [Sh82c] he gives many new characterizations of and interesting results related to properties M and β, "perfect" probability measures, marginal and conditional distributions, and simultaneous extensions of measures. For example, it is shown in [Sh82a] that Y has property β if and only if it is true that for every pair of spaces X,Z and every pair P_{xy} and P_{yz} of probability laws on $X \times Y$ and $Y \times Z$, respectively, with common marginal P_y on Y, it follows that there is a probability law P on $X \times Y \times Z$ with marginals P_{xy} and P_{yz}.

Section V: Sierpinski sets and property σ.

A set of real numbers X has property S if it intersects every set of Lebesgue measure zero in a countable set. An uncountable set with this property can be constructed (under CH) by well-ordering the G_δ sets of measure zero into a transfinite sequence $K_0, K_1, \ldots, K_\alpha, \ldots\ \alpha < \Omega$, and then (when possible) picking just one element from each set $K_\alpha - \bigcup_{\beta<\alpha} K_\beta$ (see [Ku66, p. 523]). Such a set was first constructed by Sierpinski [Si24]. We say a separable metric space X has property σ if every F_σ set in X is a G_δ in X (note that this is equivalent to X having Baire Order ≤ 1). That $S \to \sigma$ for subsets of R was first shown in [Sz 30b]. Since an uncountable set with property S can be transformed homeomorphically into a set of measure zero (which would not have property S), but property σ is clearly a topological property which would be preserved under homeomorphisms, it is clear that CH implies $S \not\leftarrow \sigma$ for subsets of R.

The sets with property S and the analogies that exist between this property and property L (rel R) were studied extensively by Sierpinski in [Si 34a, Ch III]. The property is obviously hereditary and preserved under countable unions. Sierpinski gave the following mapping theorems which characterize the property (under CH): (1) (without CH) if $f: R \to R$ is measurable, then f transforms S sets into AFC sets (defined below), and conversely, (2) (assuming CH) if $A \subseteq R$ and every measurable $f: R \to R$ transforms A into a 1st category set, then A must have property S. These results are analogous to the facts that (1') (without CH) if $f: R \to R$ is a B_w function, then f transforms sets with property L(rel R) into β sets (see [Ro38]), and (2') (assuming

CH) if $A \subseteq R$ and every B_W $f:R \to R$ transforms A into a set of measure zero, then A has property $L(\text{rel } R)$ (the proof of this would be the same as that of (2)).

Property S (rel X) can be suitably defined for $X = R^n$ or any other locally compact group, where Haar measure is available. However, it is obvious that the product of two sets with property S (rel R) will fail to have property $S(\text{rel } R^2)$. We know of no results about vector sums of Sierpinski sets such as were established about Lusin sets, and which might show, for example, that the product of two sets with property S might not have property λ or AFC (discussed below).

We do not know of very many references to papers concerning the property σ. It is obviously hereditary and preserved under homeomorphisms. We do not know if the property is preserved under continuous mappings, taking finite or countable unions, or taking products. We do not know in general if the property "Baire order $\leq \alpha$" is preserved under these operations (the countably increasing union of spaces with Baire Order ≤ 2 will have Baire order ≤ 3 [BrGa79]). It was shown in [MzSz37] that sets having property σ and finite dimension have dimension zero. This fact, plus the theorems of Hilgers (see [Ku66, p. 302]), plus CH, imply that property σ is not preserved under mappings by 1-1 functions with continuous inverses.

Section VI Rarified sets - properties λ and λ'.

A set A is said to be "rarified" or to have <u>property λ</u> if every countable subset of A is G_δ relative to A. A subset A of a space X is said to have <u>property λ' (rel X)</u> if for every countable subset C of X, $A \cup C$ has property λ. It is clear that for subsets of R,

$$(6.1) \quad \text{countable} \longrightarrow S \begin{array}{c} \nearrow \sigma \longrightarrow \searrow \\ \searrow \lambda' \text{ (rel } R) \nearrow \end{array} \lambda,$$

and that for separable metric spaces the implications not involving S hold (where R is replaced by X).

An uncountable example of a set with property λ' (rel R) was given (without CH) in [Si45b] (see [Ku 66, p. 521]).

In [Ro 39] (also see [Si 39] and [Po 55]) an example was given (without CH) of a subset M of the irrationals such that M has property λ but $M \cup \{\text{rationals}\}$ does not. This shows (without CH) that λ' (rel R) $\not\leftarrow \lambda$ for subsets of R, and that property λ is not preserved under taking even finite unions. On the other hand, property λ'(rel X) is countably additive [Si 37a].

It was shown in [Ku 33] that property λ is preserved under transformation by 1-1 functions f such that f^{-1} is continuous. From this it follows (as it did for property β) that (1) the graph of an arbitrary function with domain a λ set also has property λ, (2) any set which has the same cardinality as that of a λ set is the 1-1 continuous image of a λ set, and (3) (assuming CH) for every n, there will exist λ subsets of R^{n+1} of dimension n. (1) and (2) were pointed out in [Ku 33] and (3) was used in [MzSz37] to show (under CH) that $\sigma \not\leftarrow \lambda$.

By contrast, it was shown (under CH) in [Si 45a] that property λ' (rel R) is not even preserved under homeomorphic transformation onto another subset of R. However, if f is a homeomorphism of a space X onto a space Y, f transforms the sets with property λ' (rel X) onto sets with property λ' (rel Y). Sierpinski did show in [Si 45a] that if $A \subseteq R$ has property λ' (rel R) and is the 1-1 projection of the subset H of R^2 (i.e. H is the graph of an arbitrary real valued function with domain A), then H has property λ' (rel R^2). But we still do not know if there are subsets of R^{n+1} of dimension n with property λ' (rel R^{n+1}).

Mauldin gave an extremely powerful example in [Mn 77] which shows (under a set theoretic assumption weaker than CH) that $\sigma \not\leftrightarrow \lambda$. The example there has property λ, but has <u>Baire order $= \omega_1$</u>.

We know of no results concerning vector sums or products of spaces with properties λ or λ'.

There is one interesting characterization of property λ' in terms of concentrated sets. A subset A of a space X has property λ' (rel X) if and (under CH) only if A contain no uncountable subset which is con(rel X) [Si 45a].

Section VII: Always first category sets.

A set A will be said to have property AFC if every dense in itself subset of A is first category relative to itself. Lusin described (under CH) an uncountable subset of R with this property in his early paper [Lu 14]. He established the existence of such a set (without CH) in [Lu 21]. (Also see [Si 34b]). He showed $\lambda \longrightarrow \text{AFC}$ and (under CH) that $\lambda \not\leftarrow \text{AFC}$ in [Lu 33]. Rotherberger showed (without CH) in [Ro 39] that $\lambda \not\leftarrow \text{AFC}$.

This property is related to the sets with property B_r in the same way that the sets with property β are related to those with property M. First we review some facts concerning properties B_w and B_r (see [Ku 66] for details). If X is a complete space, the classes $B_w(\text{rel } X) \supseteq B_r(\text{rel } X)$ form σ-algebras containing the Borel sets, are closed under operation (A), and therefore contain the analytic subsets of X. If A has property B_r relative to any complete space X in which A can be embedded, then A has property B_r relative to every space in which A can be embedded (this "universal" property will be called property B_r).

There is an abundance of literature studying the equivalence between measure and Baire category, where (1) the sets of Lebesgue measure zero, (2) the Lebesgue measurable sets, (3) the universally measurable sets, and (4) the β sets are shown to be in many ways analogous to (1') the first category sets, (2') the sets with property B_w (rel X), (3') the sets with property B_r (rel X) and (4') the AFC sets, respectively. The reader is referred to the papers of Spilrajn-Marczewski and Morgan which appear in the list of references for more complete discussions of these analogies.

We know (see [Ku 66]) that for subsets A of a space X, property AFC is equivalent to each of the following:

(γ') every subset of A has property B_r, and

(δ') A is TI and has property B_r.

The class of AFC subsets of a space X is a σ-ideal (see [Ku 66] and [Si 34a]). Property AFC is preserved under homeomorphisms but (assuming CH) not under 1-1 functions with continuous inverses [Lu33]. More specifically, defining a 1-1 bimeasurable function f for which f is in Baire's class α and f^{-1} is in Baire's class β to be a <u>generalized homeomorphism of class α,β</u>, Sierpinski [Si 34d] showed that property AFC was preserved under generalized homeomorphisms of class $0,\beta$ for every $\beta < \Omega$, but (assuming CH) not necessarily under one of class 1,0. There exists a continuous $f:R \to R$ which transforms an AFC set into a non-B_r set, and there exists a continuous $f:R \to R$ for which there is an AFC set M such that $f^{-1}(M)$ is non-B_r (see [Si 37b]).

Again, we know of no results concerning vector sums of sets with property AFC. The question of whether property AFC is preserved under taking products was raised by Szpilrajn-Marczewski in "Probleme 68" at the end of Vol. 25 (1935) of <u>Fund. Math</u>. We know of no solution to the problem. Sierpinski [Si 34c] gave an example (under CH) of an AFC subset A of $I = [0,1]$ such that $A \times I$ does not have property B_r.

Recall that the subsets of R which have property B_w (rel R) are the sets whose symmetric difference with some Borel set is first category. Sierpinski [Si 34f] showed that the B_r subsets of R are not just the sets <u>(B_r^0)</u> whose symmetric difference with some

Borel set has property AFC, by showing (under CH) that there exists a collection with cardinality 2^c of B_r sets, no two of which have symmetric difference an AFC set. The result was improved in [GrRy 80], where it was shown (without CH) that there will actually exist an analytic set which does not have property B_r^0.

There is some limit to the analogies that exist between properties M and β and properties B_r and AFC. It was shown in [Sz 37b] that a subset A of R has property M (resp. property β) if and only if every homeomorphic image of M in R is Lebesgue measurable (resp. of Lebesgue measure zero). Morgan [Mo 79] recently showed (without CH) that (1) there exists a set $A \subseteq R$ which is not B_r, but every homeomorphic image of A in R is B_w (rel R) and (2) there exists a set $B \subseteq R$ which is not AFC, but every homeomorphic image of B in R is 1st category.

Section VIII: Property (s^0) and totally imperfect sets.

In [Sz 35] Szpilrajn-Marczewski defined two properties, (s) and (s^0) for subsets of complete spaces, which exhibit analogies with the pairs of properties M and β and B_r and AFC, respectively. A subset A of a complete space X is said to have property (s) (rel X) if every perfect subset P of X contains a perfect subset Q such that either $Q \subseteq A$ or $A \cap Q$ is empty. The meaning of property (s)(rel X) is made clearer by the following: a subset A of a complete space X fails to have property (s) (rel X) if and only if there exists a perfect set $P \subset X$ such that $P \cap A$ is half of a "Bernstein subdivision" of

P, i.e. both $P \cap A$ and $P - A$ are TI. It is shown in [Sz 35] that the class of subsets of a complete space X with property (s)(rel X) forms a σ-algebra which is invariant under operation (A). If X and Y are complete spaces, A has property (s) (rel X), $B \subseteq Y$, and f is a 1-1 bimeasurable transformation from A onto B, then B has property (s)(rel Y). The conclusion of this theorem fails if f is just 1-1 with continuous inverse. Note that the previous theorem implies that property (s) is an "intrinsic invariant with respect to complete spaces" [Ku 66, p. 430], as are properties M and B_r. If X and Y are complete, $A \subseteq X$ and $B \subseteq Y$, then $A \times B$ has property (s) (rel $X \times Y$) if and only A has property (s) (rel X) <u>and</u> Y has property (s) (rel Y).

A subset A of a complete space has <u>property (s^0) (rel X)</u> if every subset of A has property (s) (rel X). This requirement is equivalent to each of the following: (1) every perfect subset P of X contain a perfect set Q such that $Q \cap A$ is empty). (2) A is TI and has property (s) (rel X). The class of all subsets of a complete space X which have property (s^0)(rel X) forms a σ-ideal. If X and Y are complete spaces $A \subseteq X$ and $B \subseteq Y$ then (1) if A has property (s^0) (rel X) and f is a 1-1 function from A onto B such that f^{-1} is a Borel function, then B has property (s^0) (rel Y), and (2) $A \times B$ has property (s^0) (rel $X \times Y$) if and only if A has property (s^0) (rel X) <u>and</u> B has property (s^0) (rel Y). It is shown that for complete spaces X,

$$(8.1) \qquad \begin{array}{l} B_r(\text{rel } X) \searrow \\ \qquad\qquad\qquad (s)\ (\text{rel } X) \\ M(\text{rel } X) \nearrow \end{array}$$

and

$$(8.2) \qquad \begin{array}{l} \text{AFC} \searrow \\ \qquad\qquad (s^0)\ (\text{rel } X) \longrightarrow \text{TI}. \\ \beta \nearrow \end{array}$$

CH implies the existence of a subset of R (the union of a Lusin set and a Sierpinski set) which has property (s^0) (rel R) but is neither Lebesgue measurable nor B_r. It follows from previous results about properties β and AFC that CH implies that neither property (s) (rel X) nor property (s^0) (rel X) nor property TI is preserved under 1-1 continuous transformations, nor are they preserved under continuous transformations from all of X into X.

A final result from [Sz 35] we will mention is the following: A subset A of a complete space X has property (s^0) (rel X) if and only if $A \cup B$ is TI for every TI subset B of X.

Much of the theory concerning analogies that hold between the property pairs M - β, B_r - AFC, and (s) - (s^0) has been unified into a single theory recently in a sequence of interesting papers by John C. Morgan [Mo 76], [Mo 77], and [Mo 78].

Property TI is clearly not even finitely additive because any uncountable complete space X can be subdivided into two Bernstein sets see [Ku 66, p. 514]. However, as we indicated above, the union of a TI subset of a complete space X and an (s^0) (rel X) set will be TI. The property is preserved under

1-1 functions with continuous inverses [Ku 66, p. 519], and the consequences concerning arbitrary graphs, 1-1 continuous images, images under continuous functions from R into R, and dimensionality which follow from this theorem hold for property TI as they did for properties λ and β. The property is obviously preserved under taking products.

Section IX: Interrelationships between the Lusin branch and the Sierpinski branch.

We have already considered the questions of how categorically massive it is possible for the sets of the Lusin branch to be and of how measure theoretically massive it is possible for the sets in the Sierpinski branch to be. We now consider questions of the opposite nature. Is it possible for an uncountable set to have various properties in the Sierpinski branch and the Lusin branch simultaneously? We limit our discussion to the following properties for subsets of R:

(9.1)
$$\begin{array}{ccccc} & \nearrow & \lambda' & \searrow & \\ S & & & & \lambda \longrightarrow AFC \\ & \searrow & \sigma & \nearrow & \end{array}$$

and

(9.2)
$$\begin{array}{ccccc} & \nearrow & \text{con} & \searrow & \\ L \rightarrow \nu \rightarrow L_1 \rightarrow C(\nu) \rightarrow P & & & & C \rightarrow \beta. \\ & \searrow & C'' & \nearrow & \end{array}$$

We start with the weakest of the properties in the Sierpinski branch, property AFC. It was shown in [Si 34e] (without CH) that (β and AFC) is possible, in [Si 34a, p. 68] (with CH) that (C and AFC) is possible, and in [FrTa 80] (with CH) that

(P and AFC) is possible. We will see that CH actually implies that (C(ν) and AFC) is possible. Lusin [Lu 33] showed that there is a continuous 1-1 function f from the set Q of irrationals in [0,1] into Q which transforms every Lusin subset of Q onto an AFC set. CH implies the existence of an uncountable Lusin subset M of Q, and f(M) will be (C(ν) and AFC).

On the other hand, it can easily be shown that both (L_1 and AFC) and (C(ν) and λ) are impossible. In fact, proceeding with consideration of property λ, it is clear that (P and λ) is impossible [Si 38]. But Rothberger showed (using CH) in [Ro 41] that (con and λ) is possible. We do not know if Rothberger's example has property C", or whether (C" and λ) is even possible.

It is clear from the characterization of property λ' in terms of concentrated sets given in Section VI that (con and λ') is not possible.

The next stage would be to determine whether σ (or λ') and C" (or C or β) is possible. We do not know how this will turn out. It is clear that (β and S) is impossible.

Section X: Some Applications.

The theory of the singular sets discussed in this paper has been used extensively in the study of the properties of functions of a real variable. In fact, the first AFC set, constructed by Lusin (under CH) in his early paper [Lu14], was used to give a counterexample to the converse to Baire's theorem to the effect that every Borel measurable function $f:X \to R$ has the property that if $A \subseteq X$, then there exists $B \subseteq A$, first category relative to A, such that $f|(A - B)$ is continuous (see [Ku 66, p. 403]). Many of these applications to the study of properties of real functions are discussed in [Si 34a].

The first author's original interest in the study of these singular spaces was due to the role ν sets and L_1 sets played in an extension of Blumberg's theorem about continuous restrictions of arbitrary real functions. Blumberg [Bl 22] proved the following holds for $X = R^2$,

(B) for every $f:X \to R$ there exists $D \subseteq X$, D dense in X, such that $f|D$ is continuous.

Bradford and Goffman [BdGo 60] proved that (B) holds for a separable metric space X if and only if X is BC (i.e. no open subset of X is first category). The set D in (B) cannot be uncountable even if $X = R$. In [Br 71], Brown considered the following strengthened version of (B):

(B+) for every $f:X \to R$, there exists an uncountably dense in X set $W \subseteq X$ and a dense in W set $D \subseteq W$ such that $f|W$ is continuous at each element of D.

It was shown that (B+) holds for a separable metric space X if and (assuming CH) only if no open subset of X is the union of a first category set and a ν set.

In studying a "differentiability" version of Blumberg's theorem, Ceder [Ce 69] showed that if X is an uncountable subset of R, then the following holds:

(C) for every $f:X \to R$ there exists $D \subseteq X$, D bilaterally dense in itself, such that $f|D$ is differentiable (infinite derivatives must be allowed).

Then, in [Br 74], Brown considered the following strengthened version of (C):

(C+) for every $f:X \to R$ there exists $W \subseteq X$, W bilaterally uncountably dense in itself, and a dense subset D of W, such that $f|W$ is differentiable at each element of D.

It is shown in [Br 74] (under CH) that (C+) holds for $X \subseteq R$ if and only if X is not an L_1 set.

The authors found the theory of singular spaces to be a useful tool in studying certain completeness properties of the space P(X) of probability measures on X, where P(X) is endowed with the weak* topology. It is known (see [Pa 67]) that P(X) is separable and metrizable if and only if X is, and that P(X) is compact (or topologically complete) if and only if X has the same property. [Br 77a] and [BrCo 81] unravel the relationships that exist between requirements that X or P(X) have various completeness properties: BC, SBC, and PC (a space is "pseudo complete" if it contains a dense topologically complete subspace). A ν space was used in [Br 77a] as an example to show (under CH) that (P(X) BC)$\not\leftrightarrow$(X BC) (the reverse implication holds). This example is improved in [BrCo 81], where an SBC C" space X is used to show (under CH) that (P(X) BC)$\not\leftrightarrow$(X SBC).

Finally, we mention some recent results that indicate that the theory of singular spaces might be involved in the characterization of the spaces in which Prohorov's theorem holds. X is said to be a Prohorov space if every compact (in $P(X)$) set of probability measures has a compact set in X which nearly supports those measures. Preiss [Pr 73] showed that if X is a coanalytic subset of some complete space, then X is Prohorov if and only if X is topologically complete (TC). Then he used (under CH) a "co-P" subspace X of the unit interval I (i.e. $I-X$ was P) as an example of a (non co-analytic) Prohorov space which is not TC. When surveying this result (among many others) Topsøe [To 74] used a co-ν space, which works equally well. However, in an attempt to topologically characterize Prohorov spaces, Cox [Co 81c] went in the other direction, showing that co-C" spaces suffice for this example, and expanded on the C" cover as follows. The mutually exclusive pair of sets (X,B) in the compact metric space Z has property S means that if for each compact set $K \subseteq B$, $\{W(K,i)\}_{i=1}^{\infty}$ is a sequence of open sets containing K, and for each compact set $H \subseteq X$, U_H is an open set containing H, then there exists a compact set $H \subseteq X$ and a sequence $\{K_i\}$ of compact sets in B such that $Z = \bigcup_{i=1}^{\infty} W(K_i,i) \cup U_H$.

It is shown there that if (X,B) has property S, then X is Prohorov. Furthermore, it is clear that property S is a unifying concept between the σ-compact sets B (i.e., the topologically complete X) and the C" sets B (i.e., the co-C" sets X).

In connection with [Br 77a] and [BrCo 81], Preiss showed that if X is a Prohorov space, then X is a Baire space, but we do not know the category of $P(X)$.

FINAL COMMENT.

The authors apologize for any misinterpretations we might have placed on any of the results we have discussed. We also apologize for any oversights which may have occurred in our literature search in the preparation of this paper. It is interesting to note that of the 100 references we have listed which are related to this area of research, 45 appeared prior to 1946 and 43 have appeared since 1970.

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