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On Darboux Asymmetry

Darboux points of a real function of a real variable were investigated by many authors, for references see [1,2], but I have not found a characterization of a set of points which are Darboux points from exactly one side of an arbitrary function. This article gives a characterization of that set. First we give the definition of Darboux points.

We shall denote by $L(f,x)(L^-(f,x), L^+(f,x))$ the set of all (all left-sided, right-sided) limit points of a function f at a point x.

The function $f : R \to R$ has a Darboux point at x_0 (x_0 is a left-sided Darboux point of the function f) from the left side if

(i) $$f(x_0) \in L^-(f,x_0),$$

and

(ii) for every $\delta > 0$ and $c \in R$ such that

$$c \in (\liminf_{x \to x_0-} f(x), \limsup_{x \to x_0-} f(x))$$

there exists $x \in (x_0 - \delta , x_0)$ such that $f(x) = c$.

In an analogous way one can define right-sided Darboux points. A point x_0 is a Darboux point of a function f if it is a left-sided and right-sided Darboux point.

A point x_0 is a point of Darboux asymmetry of a function f if it is a left-sided Darboux point and is not a right-sided

Darboux point or, conversely, it is a right-sided Darboux point but is not a left-sided Darboux point of this function.

Theorem on Darboux asymmetry. _For an arbitrary function_ $f : R \to R$ _the set of Darboux asymmetry points is at most denumerable_.

Proof. Clearly, it is sufficient to prove that the set A consisting of points which are left-sided Darboux points of f but not right-sided Darboux points is at most denumerable.

If $x_0 \in A$ then there are two possibilities

(iii) $$f(x_0) \notin (\liminf_{x \to x_0+} f(x), \limsup_{x \to x_0+} f(x))$$

or

(iv) there exist $\delta > 0$ and $c \in R$ such that

$$c \in (\liminf_{x \to x_0+} f(x), \limsup_{x \to x_0+} f(x))$$

and $f(x) \neq c$ for $x \in (x_0, x_0 + \delta)$.

The set A_0 of those points which fulfill the condition (iii) is at most denumerable (see Young's theorem on asymmetry, [5]).

Let A_n $n = 1, 2, \ldots$ denote the set of those points $x_0 \in A$ for which the condition (i) is fulfilled and there exist numbers $\delta_x \geq \frac{1}{n}$ and c_x such that the condition (iv) is fulfilled and

(v) $$\min (|c_x - a_x| , |c_x - b_x|) \geq \frac{1}{n} ,$$

where

$$a_x = \liminf_{x \to x_0+} f(x) , \quad b_x = \limsup_{x \to x_0+} f(x)$$

if these limits are finite. (If one of them is infinite, then we omit the condition (v).)

Thus

$$A = \bigcup_{n=0}^{\infty} A_n .$$

Now we shall prove that each of sets A_n, n = 1, 2, ... fulfills the condition: A_n does not contain any of its left-sided points of accumulation.

Let us suppose that it is false, i.e., there exists a sequence (x_k) and $x_0 \in R$ such that

$$x_k < x_0, \quad x_k \in A, \quad x_0 \in A, \quad x_k \to x_0 .$$

Since $x_0 \in A_n$, $f(x_0) \in L^-(f,x_0)$, and according to properties proved in [3]

$$\mathcal{L}^-(f,x_0) = \mathop{\ell s}_{x < x_0} \mathcal{L}^*(f,x) \supset \mathop{\ell s}_{x< x_0} \mathcal{L}(f,x) \supset \mathop{\ell s}_{k} \mathcal{L}(f,x_k) ,$$

where $\mathop{\ell s}_{t \in T} F_t$ denotes the upper topological limit ([4]) and

$$\begin{aligned} \mathcal{L}(f,x) &= \{x\} \times L(f,x) , \\ \mathcal{L}^-(f,x) &= \{x\} \times L^-(f,x) , \\ \mathcal{L}^*(f,x) &= \{x\} \times L^*(f,x) , \\ L^*(f,x) &= L(f,x) \cup \{f(x)\}. \end{aligned}$$

Since each of sets $\mathcal{L}(f,x_k)$ has a diameter greater than $\frac{2}{n}$, the set $\mathcal{L}^-(f,x_0)$ also has diameter not less than $\frac{2}{n}$.

Assume now that $\liminf_{x \to x_0^-} f(x) = a'_x > -\infty$. Then let

$$a = \inf \{y \in R \mid (x_0,y) \in \mathop{\ell s}_{k} \mathcal{L}(f,x_k)\}.$$

Of course $a \geq a'_x > -\infty$. The set $\mathop{\ell s}_{k} \mathcal{L}(f,x_k)$ has diameter not less than $\frac{2}{n}$ and there exists k_n such that

$$\min L(f,x_k) > a - \frac{1}{n} \quad \text{and} \quad x_0 - x_{k_n} < \frac{1}{n} .$$

Then we infer that

$$c_{x_{k_n}} \in (\liminf_{x \to x_0^-} f(x), \limsup_{x \to x_0^-} f(x))$$

and

$$f(x) \neq c_{x_{k_n}} \quad \text{for} \quad x \in (x_{k_n}, x_0) .$$

This shows that x_0 is not a left-sided Darboux point of f, which is a contradiction.

One can obtain a contradiction in a similar way in a case if $\limsup_{x \to x_0^-} f(x) < \infty$.

Now if $\liminf_{x \to x_0^-} f(x) = -\infty$ and $\limsup_{x \to x_0^-} f(x) = \infty$ then there exists x_{k_n} such that $|x_{k_n} - x_0| < \frac{1}{n}$ and we obtain a contradiction as well for there exists $c_{x_{k_n}}$ such that

$f(x) \neq c_{x_{k_n}}$ for $x \in (x_{k_n}, x_{k_n} + \frac{1}{n})$ and x_0 is not a left-sided Darboux point of f.

This completes the proof that A is a denumerable union of denumerable sets.

The following example completes the theorem.

Let (x_n) be an arbitrary denumerable set in R and (α_n) a sequence of positive real numbers for which the series $\sum_{n=1}^{\infty} \alpha n$ is convergent.

Let

$$f(x) = \sum_{x_n < x} \alpha_n .$$

This function is continuous at every point $x = x_n$, $n = 1, 2, \ldots$ and it is unilaterally continuous at every x_n at which it has a Darboux point from left but not from right.

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