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A DARBOUX PROPERTY FOR TRANSFORMATIONS

Let X be a euclidean space with metric ρ, X^* a separable metric space with metric ρ^* and $\mathcal{B}$ a topological base of connected sets for X such that any translation of any set in $\mathcal{B}$ is still in $\mathcal{B}$. $f: X \to X^*$ is said to be Darboux [$\mathcal{B}$] if $f(\tilde{U})$ is connected for every $U \in \mathcal{B}$, whenever $\tilde{U}$ is a set such that $U \subset \tilde{U} \subset \overline{U}$.

Being motivated by the work in [1], the authors obtain in [5] a local characterization of Darboux transformations and a necessary and sufficient condition for a Baire type 1 transformation to be Darboux.

Theorem 1. Let X^* be a euclidean space. Then $f: X \to X^*$ is Darboux [$\mathcal{B}$] if and only if at every $x_0 \in X$, the following hold:

(i) If $U \in \mathcal{B}$ and $x_0 \in \overline{U}$, then $f(x_0) \in \overline{f(U)}$.

(ii) If $U \in \mathcal{B}$ and $x_0 \in \overline{U}$, then either $f|U \cup \{x_0\}$ is continuous at x_0 or there is a connected set $K^* \subset f(U)$ such that

$$\bigcap_{n=1}^{\infty} \overline{f(S_n(x_0) \cap U)} \subset \overline{K^*},$$

where $S_n(x_0) = \{x \in X: \rho(x, x_0) < 1/n\}$.

This generalizes a theorem in [2]. The conditions at $x_0 \in X$ are obviously necessary. The sufficiency of the conditions is proved by contradiction. A part for the proof of Theorem 1 in [1] is used.

Theorem 2. Let $f: X \to X^*$ be a Baire type 1 transformation. Then f is Darboux [$\mathcal{B}$] if and only if the property (Z), which is an analogue of a property given by Zahorski [6], is satisfied.

(Z) If V^* is open in X^* and $x_0 \in f^{-1}(V^*)$, then

$$U \cap f^{-1}(V^*) - \{x_0\} \neq \emptyset \text{ for every } U \in \mathcal{B} \text{ with } x_0 \in \overline{U}.$$

The necessity follows from the definition. The sufficiency can be obtained with the aid of Theorem 2 in [1].

A result concerning approximately continuous transformations by Goffman and Waterman [3] and a result on the derivatives of interval functions by Neugebauer [4] may follow from Theorem 2 above as special cases.

References

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