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SOME EXAMPLES ON CONTINUOUS RESTRICTIONS

For a given function $f : R \to R$, how "large" or "thick" can a subset A be for which the restriction of f to A, $f|A$, is continuous? If f is nice enough topologically, in particular, of Baire class α, then A may be taken to be residual. In fact, it is known that [5]: f <u>has the property of Baire</u> (i.e., the inverse image of any open set is the symmetric difference of an open set and a first category set) <u>if and only if there exists a residual set</u> A <u>such that</u> $f|A$ <u>is continuous</u>. On the other hand, surprisingly enough, <u>if</u> $f : R \to R$ <u>is an arbitrary function, then there exists a countable, dense subset</u> A <u>of</u> R <u>such that</u> $f|A$ <u>is continuous</u> (Blumberg [1]). The set A here cannot be taken to be uncountable, in general.

It is then natural to ask the following question:

Are there "nice" kinds of functions, f, not having the property of Baire, for which there exists a dense subset A of R with A uncountable such that f|A is continuous?

The main purpose of this article is to show that two likely candidates for this property, namely the Lebesgue measurable functions and the connected functions (i.e., the graph is a connected set), fail to have this property.

Finally we look at a somewhat related question: can a given function $f:\mathbb{R}\to\mathbb{R}$ be decomposed into countably many continuous functions, i.e., do there exist countably many disjoint sets $\{A_n\}_{n=1}^{\infty}$ whose union is $\mathbb{R}$ such that $f|A_n$ is continuous for each n? Davies [3] has constructed a semi-continuous function which cannot be decomposed into countably many functions. We improve this example by showing that such a function can also be approximately continuous.

__Notation__ __and__ __Terminology__. For a planar subset A and an interval J of $\mathbb{R}$, we define $A|J = \{(x,y) \in A : x \in J\}$. We will identify a function with its graph. We denote the cardinality of $\mathbb{R}$ by c, and will also identify the first ordinal equinumerous with $\mathbb{R}$ by c, the context will distinguish between the two uses. For a set A, $|A|$ denotes the cardinality of A. A set A is c-dense in $\mathbb{R}$ if $|A \cap 0| = c$ for each open set $0 \neq \emptyset$.

The following example appears in Kuratowski [5]. We repeat it here because we need the ideas in its proof for Examples 2 and 3.

__Example__ 1. __There__ __exists__ __a__ __function__ $f:\mathbb{R}\to\mathbb{R}$ __such__ __that__ $f|A$ __is__ __discontinuous__ __for__ __each__ __set__ A __with__ __cardinality__ c.

__Proof__. First note that if $f|A$ is continuous then, defining $g(x) = \limsup_{z\to x} f(z)$, g becomes an upper-semi-continuous function defined on A such that $f = g$ on A. Therefore, if $f|A$ is continuous then there exists a Baire 1 function defined on R containing $f|A$.

Since the class of Baire 1 functions has cardinality c we may well-order it as $\{g_\alpha\}_{\alpha<c}$. Also well-order R as $\{r_\alpha\}_{\alpha<c}$. By transfinite induction on c we define f by

$$f(r_\alpha) \in R - \{g_\beta(r_\xi) : \beta,\xi \leq \alpha\}.$$

For β fixed we have $f(r_\alpha) \neq g_\beta(r_\alpha)$ whenever $\beta < \alpha$. Therefore $\{x : g_\beta(x) = f(x)\} \subset \{r_\xi : \xi \leq \beta\}$. Therefore $|\{x : g_\beta(x) = f(x)\}| \leq |\{r_\xi : \xi \leq \beta\}| = |\beta| < c$. Hence, f agrees with each Baire 1 function on a set of cardinality less than c. So if $f|A$ is continuous, then $|A| < c$.

By Lusin's theorem a measurable function agrees with a continuous function on a set of positive measure. Hence, there exists an uncountable set A upon which the restriction is continuous. However, this uncountable set cannot be taken to be c-dense in R as shown by the next example. Observe that although a measurable function f agrees with a Baire 2 function g almost everywhere, the residual set A such that $g|A$ is continuous may coincide with $\{f \neq g\}$.

<u>Example</u> 2. <u>There exists a measurable function</u> $f : R \to R$ <u>such that</u> $f|A$ <u>is discontinuous for each</u> A <u>which is</u> c-<u>dense in</u> R.

<u>Proof</u>. Choose E to be a G_δ set dense in R and having measure zero. Then we may express R − E as $\bigcup_{n=1}^{\infty} C_n$ where each C_n is closed and nowhere dense and $C_n \cap C_m = \emptyset$ when $m \neq n$.

Since E is a complete metric space, we may use the construction in Example 1 to find a function $g : E \to (-\infty, 0)$ such that $g|A$ is discontinuous for each uncountable set $A \subset E$. Now define f by

$$f(x) = \begin{cases} g(x) & \text{if } x \in E \\ 1 - 1/n & \text{if } x \in C_n . \end{cases}$$

Obviously f is measurable. Suppose A is c-dense in R. If $|A \cap E| = c$ then $f|A \cap E = g|A \cap E$ is discontinuous. Hence, $f|A$ is discontinuous. So let us assume that $|A \cap E| < c$. Then $A - E$ is c-dense in R. Find n and x such that $x \in (A - E) \cap C_n$. Then we may select a sequence $\{z_k\}_{k=1}^{\infty}$ such that if $z_k \in C_{n_k}$, then $\{n_k\}$ is strictly increasing, and $z_k \to x$. Obviously $f(z_k) \to 1 \neq f(x) = 1 - 1/n$. Therefore, in this case too, $f|A$ is discontinuous.

A likely candidate for a class of functions admitting a c-dense set upon which the restriction is continuous would be the class of all connected functions, a smaller class than the class of Darboux functions. However, by the following example a connected function may not even admit an uncountable set upon which its restriction is continuous.

Example 3. Assuming the Continuum Hypothesis there exists a connected function $f : R \to R$ such that $f|A$ is discontinuous for each uncountable set A.

Proof. Let $\{g_\alpha\}_{\alpha<c}$ be a well-ordering of all Baire 1 functions. By the argument in Example 1 it will suffice to construct a connected function f such that $|f \cap g_\alpha| < c$ for each α.

Let $\{r_\alpha\}_{\alpha<c}$ be a well-ordering of R, and for a planar point z let $L(z)$ be the vertical line through z.

We will construct by transfinite induction on c families of countable sets $\{B_\beta\}_{\beta<c}$ and $\{D_\beta\}_{\beta<c}$ as follows:

For notational convenience, define $B_{-1} = D_{-1} = \emptyset$. Having selected B_β and D_β for each $\beta < \alpha$, define

$$E_\alpha = g_\alpha - \bigcup\{g_\beta : \beta < \alpha\} - \bigcup\{L(z) : z \in \bigcup\{D_\beta : \beta < \alpha\}\}.$$

If $E_\alpha = \emptyset$, we put $D_\alpha = \emptyset$. If $E_\alpha \neq \emptyset$, pick D_α to be a countable dense subset of E_α such tht $D_\alpha \cap E_\alpha \cap C_\alpha$ is also dense in $E_\alpha \cap C_\alpha$ where $C_\alpha = \{(x, g_\alpha(x)) : g_\alpha$ is continuous at $x\}$. (Note that dom C_α is residual in R.)

If $r_\alpha \in \text{dom} \bigcup\{D_\beta : \beta < \alpha\}$ put $B_\alpha = D_\alpha$. If $r_\alpha \notin \text{dom} \bigcup\{D_\beta : \beta < \alpha\}$ put $B_\alpha = D_\alpha \cup \{(r_\alpha, g_\alpha(r_\alpha))\}$. Finally put

$$f = \bigcup\{B_\alpha : \alpha < c\}.$$

Clearly $B_\alpha \cap B_\xi = \emptyset$ when $\alpha \neq \xi$. It follows that f is a function. Moreover, for each α, $r_\alpha \in \bigcup\{\text{dom } D_\beta : \beta \leq \alpha\}$ so that dom f = R. In addition, for each α,

$$|f \cap g_\alpha| =$$

$$\left|\bigcup_{\xi<c}(B_\xi \quad g_\alpha)\right| = \left|\bigcup_{\xi\leq\alpha}(B_\xi \quad g_\alpha)\right| =$$

$$|\alpha|\aleph_0 = |\alpha| < c.$$

By a result in [4] it will suffice to show that f hits each continuum with domain a non-degenerate interval. Let H be a continuum with $|\text{dom } H| > 1$. For $x \in \text{dom } H$ define

$$h(x) = \lim \{\sup\{\text{rng}[H \mid (x - 1/n,\ x + 1/n)]\}\}.$$

Then h is upper-semi-continuous and $h \subset H$. Hence, there exists a g_γ such that $\text{dom}(g_\gamma \cap H)$ is somewhere dense.

Now let α be the first ξ such that $\text{dom}(g_\xi \cap H)$ is somewhere dense. Suppose $\text{dom}(g_\alpha \cap H)$ is dense in an open interval J. If $\beta < \alpha$, then $\text{dom}(g_\beta \cap H)$ is nowhere dense in J. It follows tht $\text{dom}(E_\alpha \cap H)$ is residual in J. Since $\text{dom}C_\alpha$ is residual in J the set $\text{dom}C_\alpha \cap \text{dom}(E_\alpha \cap H)$ is also residual in J. Since H is closed it follows that $C_\alpha \cap E_\alpha$ is a dense subset of $E_\alpha \cap H$. Therefore, D_α intersects H and f intersects H.

It is unknown whether the requirement of the Continuum Hypothesis can be omitted from Example 3.

<u>Example</u> 4. <u>There exists a bounded, approximately continuous, lower-semi-continuous function which cannot be decomposed into countably many continuous functions</u>.

<u>Proof</u>. By Theorem 2 of Davies [2] it suffices to construct a function f on $I = [0,1]$ to I such that (*) $f \cap A \neq \emptyset$ for each closed set $A \subset I \times I$ for which $\text{dom}A = I$. In [3] Davies constructs a lower-semi-continuous function f with property (*). Actually he shows that there exists a perfect, nowhere

dense null set J such that for each closed set A with $\text{dom}A = I$ there exists $x \varepsilon J$ such that $(x,f(x)) \varepsilon A$. Let $\{I_n\}_{n=1}^{\infty}$ be an enumeration of the components of $I - J$. On each I_n choose a continuous function g_n such that $\text{rng}\, g_n = I$ and $g_n(a) = f(a)$ if a is an endpoint of I_n. Now put

$$g(x) = \begin{cases} f(x) & \text{if } x \varepsilon J \\ g_n(x) & \text{if } x \varepsilon I_n. \end{cases}$$

Obviously g is lower-semi-continuous, Darboux, and has property (*). By Maximoff's theorem [6] there exists a homeomorphism h from I onto I such that $g \circ h$ is approximately continuous. Clearly $g \circ h$ is also lower-semi-continuous. If A is a closed set with $\text{dom}\, A = I$, then $\{(h(x),y) : (x,y) \varepsilon A\}$ is a closed set with domain I and hence, intersects g. But then $g \circ h$ intersects A. Therefore, $g \circ h$ has property * and is the desired function.

Observe that the function of Example 4 must also be a derivative.

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