Real Analysis Exchange Vol. 7 (1981-82)
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On continuous periodic extensions of functions

It is known [2] that there exist unbounded sets E of real numbers such that for any bounded real function Φ defined on E there exists a continuous periodic function f defined on the whole real line R such that $\Phi(x) = f(x)$ for any $x \in E$. We shall call a set E of this property a Marczewski set because of the name of the author of the question whether such sets exist. A partial answer to the question of Marzewski can be found in the paper [3]. Its author, J. Mycielski, proved that if $t_n^{-1} \cdot t_{n+1} > 3+\delta$, where $\delta > 0$, then any function Φ defined on the set E of the numbers t_n with the range consisting of two points has a continuous periodic extension. The following complete answer to Marczewski's question was given in the paper [2]. If

(i) $\quad t_n^{-1} t_{n+1} \geq \delta_{n+1}^{-1} (c+\delta_{n+2}), \ \delta_n > 0, \ \sum_{n=1}^{\infty} \delta_n = c < +\infty$

then the set of all numbers t_n is a Marczewski set. In this paper we extend this result to the following theorem.

Theorem 1. For any set E consisting of the numbers t_n following conditions (i) and for any bounded function Φ defined on E there exists a Lipschitz, periodic, piecewise monotone function f defined on R such that $\Phi(t_n) = f(t_n)$ for any n and the range $f(R) = [\inf \Phi, \sup \Phi]$.

We also discuss the problem of the power of a set of periods of continuous periodic extensions of a bounded function defined on a Marczewski set.

We shall limit ourselves to indicate the successive steps of the proof of Theorem 1 without detailed substantiation. In the beginning we construct a Lipschitz function ψ on the interval $[0,c]$ in such a manner that $\psi(0) = \psi(c) = \inf \Phi$, $\psi(\gamma) = \sup \Phi$, where $\gamma = 2^{-1} \delta_1 + \sum_{n=2}^{\infty} \delta_n = c - 2^{-1}\delta_1$, ψ is linear in $[\gamma, c]$, non-decreasing in $[0,\gamma]$ and constant and equal to $\Phi(t_n)$ in an interval $[d_n, d_n + \delta_{n+1}]$ contained in $[0,\gamma]$ for $n = 1, 2, \ldots$. Extend ψ to a periodic function with period c defined on the whole R. Then ψ takes the value $\Phi(t_n)$ on the intervals $[kc+d_n, kc+d_n + \delta_{n+1}]$ where k are integers. It is sufficient to show that there exists a number r_0 such that $r_0^{-1}x_n \in \bigcup_k [kc+d_n, kc+d_n + \delta_{n+1}]$. Then $f(x) := \psi(r_0^{-1}x)$ is the solution for Φ and r_0c is period of f.

Let $\theta_n(r) := x^{-1}r$. Choose for J_1 a closed interval such that $\theta_1(J_1) = [d_1, d_1+\delta_2]$ and let $L_1 := \theta_2(J_1)$. The first inequality of (i) implies that the length $|L_1| \geq c + \delta_3$. Therefore the interval L_1 contains at least one interval of the form $[kc+d_2, kc+d_2 + \delta_3]$. Fix one of them as $[k_2c + d_2, k_2c + d_2 + \delta_3] := B_2$ and choose for J_2 a closed interval such that $\theta_2(J_2) = B_2$. Of course $J_1 \supset J_2$. In a similar manner we define a decreasing sequence of closed intervals J_n and choose k_n such that $\theta_n(J_n) = [k_nc + d_n, k_nc + d_n + \delta_{n+1}]$. Let $L_n = \theta_{n+1}(J_n)$. Then $|L_n| \geq c + \delta_{n+2}$. The intersection $\bigcap J_n$ is a singleton r_0. Obviously $r_0^{-1}x_n = \theta_n(r_0) \in \theta_n(J_n)$. This ends the proof.

Let E be an unbounded set of real numbers and let Φ be a bounded function defined on E. Denote by $P(E, \Phi)$ the set of all continuous periodic extensions of Φ and by $R(E,\Phi)$ the set of all proper periods of functions belonging to $P(E,\Phi)$. Marczewski's question can be expressed as follows: Does there exist an unbounded set E such that for any bounded function $\Phi : E \to R$ the set $P(E,\Phi)$ is non-empty? Hartman [1] observed that for any unbounded E and any bounded Φ the set $R(E,\Phi)$ is a zero-set. The following theorems complete his observation.

Theorem 2. If the numbers t_n follow the conditions (i) and there exists an infinite sequence of indices n_i such that $t_{n_i}^{-1} t_{n_i+1} \geq \delta_{n_i+1}^{-1}(2c + \delta_{n_i+2})$ then for each bounded function defined on the set $E = \{t_n\}$ the set $R(E,\Phi)$ has the power of the continuum.

Proof: We continue the argumentation of the proof of Theorem 1. $|L_{n_i}| \geq 2c + \delta_{n_i+2}$. Thus each interval L_{n_i} contains at least two intervals $[k_{n_i+1}c + d_{n_i+1}, k_{n_i+1}c + d_{n_i+1} + \delta_{n_i+2}]$ and $[k_{n_i+1} + 1)\cdot c + d_{n_i+1}, (k_{n_i+1} + 1)\cdot c + d_{n_i+1} + \delta_{n_i+2}]$. There are two different intervals J_{n_i} associated with them. Denote them by $J_{n_i,0}$ and $J_{n_i,1}$. Obviously (ii) $J_{n_i,0} \cap J_{n_i,1} = \emptyset$.

Thus defining the interval J_{n_i} we have to choose one of the intervals J_{n_i,j_i} where j_i equals 0 or 1. There are as many sequences $\{n_i\}$ as there are sequences of 0's and 1's, so they form a set of the power of the continuum. $\{r_0\} = \bigcap_{n=1}^{\infty} J_n = \bigcap_{i=1}^{\infty} J_{n_i,j_i}$. So it is implied by (ii) that different sequences $\{n_i\}$ determine different numbers r_o.

Theorem 3. If E is a Marczewski set then for any bounded function Φ: $E\to R$ the set $R(E,\Phi)$ is non-enumerable.

Proof. Suppose to the contrary that there exists a Marczewski set E and a bounded function Φ_o: $E\to R$ such that $R(E,\Phi_o)$ is at most countable. Without loss of generality we may assume that $\Phi_o(E)\subset[0,1]$. Let us arrange the set $R(E,\Phi_o)$ as an infinite sequence $\{r_n\}$. Perhaps $r_n=r_m$ for some n, m. For any r_n there exists a bounded function Φ_n defined on E with the range consisting of two numbers 0 and $(n+1)^{-1}$ such that $r_n \notin R(E,\Phi_n)$. The function $\Psi(x) = \{\Phi_o(x), \Phi_1(x), \Phi_2(x),\ldots\}$ maps the set E into the Hilbert's cube. The Hilbert's cube is a Peano curve. Consequently there exists a continuous function G mapping [0,1] onto the Hilbert's cube. It is easy to prove that each Marczewski set is countable. Let us arrange the set E as an infinite sequence $\{x_n\}$. Take points $t_n\in G^{-1}(\Psi(x_n))$. Then the function $\hat{\Phi}$ defined on E by the formula $\hat{\Phi}(x_n) = t_n$ is real and bounded. So $\hat{\Phi}$ is extendable to a function $f\in P(E,\hat{\Phi})$. Let r_o be a period of f. Then the composite function $F(x) = G(f(x))$ maps R into the Hilbert's cube, is continuous and periodic r_o. Obviously $F(x_n) = \Psi(x_n)$. So F is a continuous and periodic and tension of Ψ. Each component f_n of F is continuous and periodic and is an extension of Φ_n. So $r_o \neq r_n$ for n = 1, 2, ... and (iii) $r_o \notin R(E, \Phi_o)$. The function f_o is the first component of F and consequently f_o is a continuous periodic extension of Φ_o with period r_o. Thus $r_o\in R(E,\Phi_o)$ contrary to (iii). This complete the proof.

Problems. 1. Is any Marczewski set congruent to a set E consisting of numbers t_n satisfying conditions (i)?

2. Is $P(E,\Phi)$ a set of the power of the continuum for and Marczewski set E and any bounded $\Phi : E \to R$?

3. Let E consist of numbers t_n satisfying the condition (i). Does there exist for any bounded $\Phi : E \to R$ a differentiable function belonging to $P(E,\Phi)$?

4. Let ψ be a Lipschitz function defined on $[0,c]$, constant on the intervals $[d_n, d_n + \delta_{n+1}]$. Does there exist a homeomorphism h mapping $[0,c]$ onto $[0,c]$ such that $\psi \circ h$ is differentiable and $h(d_n + \delta_{n+1}) - h(d_n) = \delta_{n+1}$?

A positive solution of the 4-th problem implies the identical solution of the 3-rd problem. Indeed, it is enough to replace ψ by $\psi \circ h$ in the proof of the Theorem 2.

REFERENCES

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